

Symmetry-breaking colimit of topological phases

joint work with Guo Chuan Thiang, [arxiv:2607.28181](https://arxiv.org/abs/2607.28181).



Yuezhao Li

Peking University



Noncommutative geometry in NYC

October 7, 2026

Goal of my talk

Let G be a discrete countable group.

We shall provide a dichotomy of the $K(O)$ -theory of the reduced group C^* -algebra / equivariant Roe C^* -algebra of G into **strong/weak** parts:

$$0 \rightarrow \mathrm{wk}K_*(C_r^*(G)) \rightarrow K_*(C_r^*(G)) \rightarrow \mathrm{sk}K_*(C_r^*(G)) \rightarrow 0.$$


This is motivated by the notion of strong/weak phases of topological insulators and their macroscopic nature.

A geometric setup for higher/coarse index

How to do index theory on non-compact manifolds?

A geometric setup for higher/coarse index

How to do index theory on non-compact manifolds?

John Roe introduced **coarse index theory** to handle this.

A geometric setup for higher/coarse index

How to do index theory on non-compact manifolds?

John Roe introduced **coarse index theory** to handle this.

Typical example

- ▶ X : complete, non-compact Riemannian manifold.
- ▶ $\Gamma \curvearrowright X$: discrete, proper, cocompact.
- ▶ $\mathcal{D}$: Γ -invariant self-adjoint Dirac-type operator.

The Γ -equivariant (coarse/higher) index of $\mathcal{D}$ is defined as a certain K-theory class

$$\mathrm{ind}_\Gamma(\mathcal{D}) \in K_*(C^*(X)^\Gamma).$$

- ▶ $C^*(X)^\Gamma$: equivariant Roe algebra.

Equivariant Roe algebra, 1

Let Γ be a discrete countable group, $\Gamma \curvearrowright X$ proper action.

A (twisted) **X - Γ -module** is a Hilbert space $\mathcal{H}$ equipped with:

- ▶ a non-degenerate $*$ -representation $\varrho: C_0(X) \rightarrow \mathbb{B}(\mathcal{H})$;
- ▶ a (σ -twisted) unitary representation $U: \Gamma \rightarrow \text{PU}(\mathcal{H})$;
- ▶ covariance relation: $U_g \varrho(f) U_g^* = \varrho(\alpha_g(f))$.

Equivariant Roe algebra, 1

Let Γ be a discrete countable group, $\Gamma \curvearrowright X$ proper action.

A (twisted) **X - Γ -module** is a Hilbert space $\mathcal{H}$ equipped with:

- ▶ a non-degenerate $*$ -representation $\varrho: C_0(X) \rightarrow \mathbb{B}(\mathcal{H})$;
- ▶ a (σ -twisted) unitary representation $U: \Gamma \rightarrow \text{PU}(\mathcal{H})$;
- ▶ covariance relation: $U_g \varrho(f) U_g^* = \varrho(\alpha_g(f))$.

Definition The (σ -twisted) equivariant Roe algebra

$$C^*(X, \mathcal{H})^{\Gamma, \sigma}$$

is the C^* -subalgebra of $\mathbb{B}(\mathcal{H})$ generated by all Γ -invariant, finite propagation and locally compact operators.

Equivariant Roe algebra, 2

Basic Setup

- X proper metric space.
- Γ discrete countable group, acting properly on X .
- $\mathcal{H}$ σ -twisted X - Γ -module
admitting a **bounded** fundamental domain.

Equivariant Roe algebra, 2

Basic Setup

- X proper metric space.
- Γ discrete countable group, acting properly on X .
- $\mathcal{H}$ σ -twisted X - Γ -module admitting a **bounded** fundamental domain.

Proposition Under the **basic setup**:

$$C^*(X, \mathcal{H})^{\Gamma, \sigma} \simeq C_r^*(\Gamma, \bar{\sigma}) \otimes \mathbb{K}(\mathcal{H}_D)$$

- ▶ $\mathcal{H}_D := \varrho(\chi_D)(\mathcal{H})$ for D a **fundamental domain** for $\Gamma \curvearrowright X$.
- ▶ $C_r^*(\Gamma, \bar{\sigma})$: twisted reduced group C*-algebra.

An abelian example

- ▶ $X = \mathbb{R}^d$, $\Gamma \simeq \mathbb{Z}^d$ a lattice of X .

An abelian example

- ▶ $X = \mathbb{R}^d$, $\Gamma \simeq \mathbb{Z}^d$ a lattice of X .
- ▶ Γ -invariant Schrödinger operator on $L^2(\mathbb{R}^d)$:

$$H := -\Delta + V, \quad \text{with } V \in L^\infty(\mathbb{R}^d) \text{ being } \Gamma\text{-invariant.}$$

An abelian example

- ▶ $X = \mathbb{R}^d$, $\Gamma \simeq \mathbb{Z}^d$ a lattice of X .
- ▶ Γ -invariant Schrödinger operator on $L^2(\mathbb{R}^d)$:

$$H := -\Delta + V, \quad \text{with } V \in L^\infty(\mathbb{R}^d) \text{ being } \Gamma\text{-invariant.}$$

- ▶ Assume that the Fermi energy $E \notin \text{spec}(H)$. Then

$$P_{\leq E} := \chi_{(-\infty, E)}(H) \in C^*(\mathbb{R}^d, L^2\mathbb{R}^d)^\Gamma$$

so defines a class in

$$K_0(C^*(\mathbb{R}^d, L^2\mathbb{R}^d)^\Gamma) \simeq K_0(C(\widehat{\Gamma})) \simeq \mathbb{Z}^{2^{d-1}}$$

An abelian example

- ▶ $X = \mathbb{R}^d$, $\Gamma \simeq \mathbb{Z}^d$ a lattice of X .

- ▶ Γ -invariant Schrödinger operator on $L^2(\mathbb{R}^d)$:

$$H := -\Delta + V, \quad \text{with } V \in L^\infty(\mathbb{R}^d) \text{ being } \Gamma\text{-invariant.}$$

- ▶ Assume that the Fermi energy $E \notin \text{spec}(H)$. Then

$$P_{\leq E} := \chi_{(-\infty, E)}(H) \in C^*(\mathbb{R}^d, L^2\mathbb{R}^d)^\Gamma$$

so defines a class in

$$K_0(C^*(\mathbb{R}^d, L^2\mathbb{R}^d)^\Gamma) \simeq K_0(C(\widehat{\Gamma})) \simeq \mathbb{Z}^{2^{d-1}}$$

- ▶ $d = 2$:

$$K_0(C(\mathbb{T}^2)) \simeq \underbrace{\mathbb{Z}}_{\text{rank of v.b.}} \oplus \underbrace{\mathbb{Z}}_{\text{Chern number}}.$$

Strong/Weak topological phases, 1

More interesting topological phases/invariants occur in TRS/PHS systems and can be described using **KO**-theory.

$$\begin{aligned} \mathrm{KO}_4(C^*(\mathbb{R}^2)^{\mathbb{Z}^2}) &\simeq \underbrace{\mathbb{Z}}_{\text{rank}} \oplus \underbrace{\mathbb{Z}/2}_{\text{Kane-Mele}}, \\ \mathrm{KO}_4(C^*(\mathbb{R}^3)^{\mathbb{Z}^3}) &\simeq \underbrace{\mathbb{Z}}_{\text{rank}} \oplus \underbrace{(\mathbb{Z}/2)^3}_{\text{weak}} \oplus \underbrace{\mathbb{Z}/2}_{\text{strong}}. \end{aligned}$$

Strong/Weak topological phases, 1

More interesting topological phases/invariants occur in TRS/PHS systems and can be described using **KO**-theory.

$$\begin{aligned} \mathrm{KO}_4(\mathrm{C}^*(\mathbb{R}^2)^{\mathbb{Z}^2}) &\simeq \underbrace{\mathbb{Z}}_{\text{rank}} \oplus \underbrace{\mathbb{Z}/2}_{\text{Kane-Mele}}, \\ \mathrm{KO}_4(\mathrm{C}^*(\mathbb{R}^3)^{\mathbb{Z}^3}) &\simeq \underbrace{\mathbb{Z}}_{\text{rank}} \oplus \underbrace{(\mathbb{Z}/2)^3}_{\text{weak}} \oplus \underbrace{\mathbb{Z}/2}_{\text{strong}}. \end{aligned}$$

- ▶ The **strong** one is well-established experimentally.
- ▶ The **weak** ones are generally considered to be non-robust.

Strong/Weak topological phases, 1

More interesting topological phases/invariants occur in TRS/PHS systems and can be described using **KO**-theory.

$$\begin{aligned} \mathrm{KO}_4(C^*(\mathbb{R}^2)^{\mathbb{Z}^2}) &\simeq \underbrace{\mathbb{Z}}_{\text{rank}} \oplus \underbrace{\mathbb{Z}/2}_{\text{Kane-Mele}}, \\ \mathrm{KO}_4(C^*(\mathbb{R}^3)^{\mathbb{Z}^3}) &\simeq \underbrace{\mathbb{Z}}_{\text{rank}} \oplus \underbrace{(\mathbb{Z}/2)^3}_{\text{weak}} \oplus \underbrace{\mathbb{Z}/2}_{\text{strong}}. \end{aligned}$$

- ▶ The **strong** one is well-established experimentally.
- ▶ The **weak** ones are generally considered to be non-robust.

Goal Find an “intrinsic” way to distinguish weak/strong phases for a general (discrete) symmetry group Γ ,
i.e. K-theory of equivariant Roe algebra/group C*-algebra of Γ .

Strong/Weak topological phases, 2

How to distinguish weak/strong phases for $\Gamma \simeq \mathbb{Z}^d$?

- ▶ **PV sequence**: via (non-canonical) splitting of Pimsner–Voiculescu sequence.

Strong/Weak topological phases, 2

How to distinguish weak/strong phases for $\Gamma \simeq \mathbb{Z}^d$?

- ▶ **PV sequence**: via (non-canonical) splitting of Pimsner–Voiculescu sequence.
- ▶ **Geometric**: via the map $\widehat{\Gamma} \simeq \widehat{\Gamma} \simeq \mathbb{T}^d \rightarrow \mathbb{S}^d$ which collapses onto the top-dimensional cell.

Strong/Weak topological phases, 2

How to distinguish weak/strong phases for $\Gamma \simeq \mathbb{Z}^d$?

- ▶ **PV sequence**: via (non-canonical) splitting of Pimsner–Voiculescu sequence.
- ▶ **Geometric**: via the map $\widehat{\Gamma} \simeq \widehat{\Gamma} \simeq \mathbb{T}^d \rightarrow \mathbb{S}^d$ which collapses onto the top-dimensional cell.
- ▶ **Coarse-geometric**: via the forgetful map $C_r^*(\Gamma) \rightarrow C^*(\Gamma)$.

Strong/Weak topological phases, 2

How to distinguish weak/strong phases for $\Gamma \simeq \mathbb{Z}^d$?

- ▶ **PV sequence**: via (non-canonical) splitting of Pimsner-Voiculescu sequence.
- ▶ **Geometric**: via the map $\widehat{\Gamma} \simeq \widehat{\Gamma} \simeq \mathbb{T}^d \rightarrow \mathbb{S}^d$ which collapses onto the top-dimensional cell.
- ▶ **Coarse-geometric**: via the forgetful map $C_r^*(\Gamma) \rightarrow C^*(\Gamma)$.

Problems:

Strong/Weak topological phases, 2

How to distinguish weak/strong phases for $\Gamma \simeq \mathbb{Z}^d$?

- ▶ **PV sequence**: via (non-canonical) splitting of Pimsner–Voiculescu sequence.
- ▶ **Geometric**: via the map $\widehat{\Gamma} \simeq \widehat{\Gamma} \simeq \mathbb{T}^d \rightarrow \mathbb{S}^d$ which collapses onto the top-dimensional cell.
- ▶ **Coarse-geometric**: via the forgetful map $C_r^*(\Gamma) \rightarrow C^*(\Gamma)$.

Problems:

- ▶ Hard to generalise to general Γ .

Strong/Weak topological phases, 2

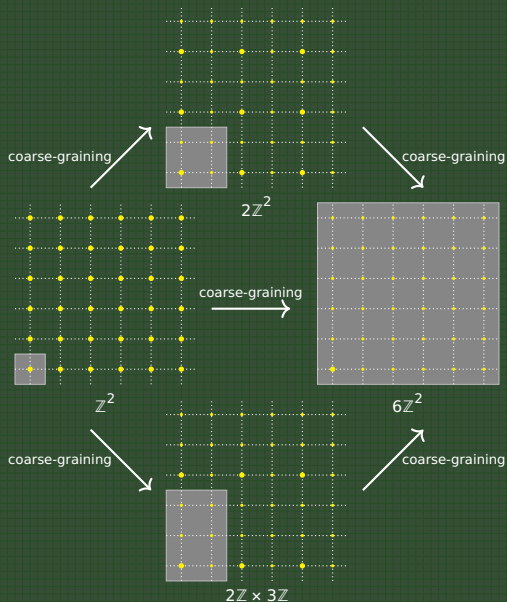
How to distinguish weak/strong phases for $\Gamma \simeq \mathbb{Z}^d$?

- ▶ **PV sequence**: via (non-canonical) splitting of Pimsner-Voiculescu sequence.
- ▶ **Geometric**: via the map $\widehat{\Gamma} \simeq \widehat{\Gamma} \simeq \mathbb{T}^d \rightarrow \mathbb{S}^d$ which collapses onto the top-dimensional cell.
- ▶ **Coarse-geometric**: via the forgetful map $C_r^*(\Gamma) \rightarrow C^*(\Gamma)$.

Problems:

- ▶ Hard to generalise to general Γ .
- ▶ Lack of physical picture.

Consistent symmetry breaking, 1



Consistent symmetry breaking, 2

- ▶ If $G \subseteq \Gamma$ is a subgroup, then there is a natural injective *-homomorphism

$$l_{G \setminus \Gamma}: C^*(X, \mathcal{H})^{\Gamma, \sigma} \hookrightarrow C^*(X, \mathcal{H})^{G, \sigma}.$$

by viewing $\mathcal{H}$ as an **X - G -module**
and restricting the group 2-cocycle $\sigma: \Gamma \times \Gamma \rightarrow U(1)$ to G .

- ▶ Such injective maps are consistently related:

$$l_{K \setminus G} = l_{K \setminus H} \circ l_{H \setminus G}, \quad K \subseteq H \subseteq G.$$

- ▶ This gives a **directed system** of equivariant Roe algebras and their K-theory.

Symmetry breaking in K-theory, 1

Definition The symmetry breaking (directed) poset is

$$\mathfrak{S} = \{\text{finite-index subgroups of } \Gamma\}$$

and $G \prec H$ iff H is a subgroup of G .

Symmetry breaking in K-theory, 1

Definition The symmetry breaking (directed) poset is

$$\mathfrak{S} = \{\text{finite-index subgroups of } \Gamma\}$$

and $G \prec H$ iff H is a subgroup of G .

The symmetry breaking Roe algebra

$$C^*(X, \mathcal{H})^{\mathfrak{S}} := \varinjlim_{G \in \mathfrak{S}} C^*(X, \mathcal{H})^G \simeq \overline{\bigcup_{G \in \mathfrak{S}} C^*(X, \mathcal{H})^G}.$$

Symmetry breaking in K-theory, 1

Definition The symmetry breaking (directed) poset is

$$\mathfrak{S} = \{\text{finite-index subgroups of } \Gamma\}$$

and $G \prec H$ iff H is a subgroup of G .

The symmetry breaking Roe algebra

$$C^*(X, \mathcal{H})^{\mathfrak{S}} := \varinjlim_{G \in \mathfrak{S}} C^*(X, \mathcal{H})^G \simeq \overline{\bigcup_{G \in \mathfrak{S}} C^*(X, \mathcal{H})^G}.$$

Consistency with symmetry breaking means that any G -dependent physical prediction must factor through this colimit:

A commutative diagram illustrating the consistency of physical predictions with symmetry breaking. The diagram consists of three nodes and three arrows:

- Top-left node: $K_*(C^*(X, \mathcal{H})^G)$
- Bottom-center node: $K_*(C^*(X, \mathcal{H})^{\mathfrak{S}})$
- Top-right node: experimental quantities

The arrows are:

- A solid horizontal arrow from $K_*(C^*(X, \mathcal{H})^G)$ to experimental quantities.
- A solid diagonal arrow from $K_*(C^*(X, \mathcal{H})^G)$ to $K_*(C^*(X, \mathcal{H})^{\mathfrak{S}})$, labeled $\iota_{\mathfrak{S} \setminus G}$.
- A dotted diagonal arrow from $K_*(C^*(X, \mathcal{H})^{\mathfrak{S}})$ to experimental quantities.

Symmetry breaking in K-theory, 2

- Let $H \subseteq G$. The symmetry breaking maps

$$l_{H \setminus G}: C^*(X, \mathcal{H})^{G, \sigma} \rightarrow C^*(X, \mathcal{H})^{H, \sigma}$$

induces a *-homomorphism $C_r^*(G, \bar{\sigma}) \rightarrow \mathbb{M}_{[G:H]} C_r^*(H, \bar{\sigma})$.

- Such induced *-homomorphism is **not** canonical: it depends on the split section $H \setminus G \rightarrow G$ and the fundamental domains for G and H .
- Nevertheless: the induced **K-theory** maps are well-defined and gives an isomorphic **directed system of abelian groups**:

$$\begin{array}{ccc} K_i(C^*(X, \mathcal{H})^G) & \xrightarrow{K_i(l_{H \setminus G})} & K_i(C^*(X, \mathcal{H})^H) \\ \alpha_G \downarrow \sim & & \sim \downarrow \alpha_H \\ K_i(C_r^*(G, \bar{\sigma})) & \xrightarrow{\phi_{H \setminus G}^{\bar{\sigma}}} & K_i(C_r^*(H, \bar{\sigma})). \end{array}$$

Example: 1D symmetry breaking, 1

Let $X = \mathbb{R}$, $\Gamma \simeq \mathbb{Z}$ a lattice in X .

- Symmetry breaking poset $\mathfrak{S} = \{m\mathbb{Z} \mid m \in \mathbb{N}\}$.

Example: 1D symmetry breaking, 1

Let $X = \mathbb{R}$, $\Gamma \simeq \mathbb{Z}$ a lattice in X .

- Symmetry breaking poset $\mathfrak{S} = \{m\mathbb{Z} \mid m \in \mathbb{N}\}$.
- The “symmetry breaking” map

$$K_i(\iota_{n \setminus m}): K_i(C^*(\mathbb{R})^{m\mathbb{Z}}) \rightarrow K_i(C^*(\mathbb{R})^{n\mathbb{Z}}), \quad m \mid n.$$

can be identified with the induced map of a certain *-homomorphism (here $N := \frac{n}{m}$).

$$\begin{aligned} C_r^*(m\mathbb{Z}) &\rightarrow C_r^*(n\mathbb{Z}) \otimes \mathbb{K}(\ell^2(n\mathbb{Z} \setminus m\mathbb{Z})) \\ &\simeq \mathbb{M}_N(C^*(n\mathbb{Z})) \end{aligned}$$

Example: 1D symmetry breaking, 1

Let $X = \mathbb{R}$, $\Gamma \simeq \mathbb{Z}$ a lattice in X .

- Symmetry breaking poset $\mathfrak{S} = \{m\mathbb{Z} \mid m \in \mathbb{N}\}$.
- The “symmetry breaking” map

$$K_i(\iota_{n \setminus m}): K_i(C^*(\mathbb{R})^{m\mathbb{Z}}) \rightarrow K_i(C^*(\mathbb{R})^{n\mathbb{Z}}), \quad m \mid n.$$

can be identified with the induced map of a certain *-homomorphism (here $N := \frac{n}{m}$).

$$\begin{aligned} C_r^*(m\mathbb{Z}) &\rightarrow C_r^*(n\mathbb{Z}) \otimes \mathbb{K}(\ell^2(n\mathbb{Z} \setminus m\mathbb{Z})) \\ &\simeq \mathbb{M}_N(C^*(n\mathbb{Z})) \end{aligned}$$

- (Lots of choices involved, though all choices lead to the same K-theory map).

Example: 1D symmetry breaking, 2

- ▶ Let w_m be the m -shift on $\ell^2(m\mathbb{Z})$.
- ▶ The map is given on generators by

$$1 \mapsto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}_{N \times N}, \quad w_m \mapsto \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ w_n & & & & \end{pmatrix}_{N \times N}$$



Example: 1D symmetry breaking, 3

► Passing to K-theory $K_i(C^*(\mathbb{R})^{m\mathbb{Z}}) \rightarrow K_i(C^*(\mathbb{R})^{n\mathbb{Z}})$:

$$\underbrace{[1_m]_0}_{K_0\text{-generator}} \mapsto N \cdot \underbrace{[1_n]_0}_{K_0\text{-generator}}, \quad \underbrace{[w_m]_1}_{K_1\text{-generator}} \mapsto \underbrace{[w_n]_1}_{K_1\text{-generator}}.$$

Example: 1D symmetry breaking, 3

- Passing to K-theory $K_i(C^*(\mathbb{R})^{m\mathbb{Z}}) \rightarrow K_i(C^*(\mathbb{R})^{n\mathbb{Z}})$:

$$\underbrace{[1_m]_0}_{K_0\text{-generator}} \mapsto N \cdot \underbrace{[1_n]_0}_{K_0\text{-generator}}, \quad \underbrace{[w_m]_1}_{K_1\text{-generator}} \mapsto \underbrace{[w_n]_1}_{K_1\text{-generator}}.$$

- Passing to the (direct) colimit:

$$K_i(C^*(\mathbb{R})^{\mathbb{G}}) \simeq \begin{cases} \mathbb{Q} & i = 0; \\ \mathbb{Z} & i = 1. \end{cases}$$

Symmetry breaking in higher dimensions, 1

How to do computation in higher-dimensional / real cases

- For $X = \mathbb{R}^d$ and $\Gamma \simeq \mathbb{Z}^d$: there is a **cofinal** subset of $\mathfrak{S}$, consisting of subgroups of $\mathbb{Z}^d$ given by

$$\{(mx_1, \dots, mx_d) : (x_1, \dots, x_d) \in \mathbb{Z}^d\}, \quad m \in \mathbb{N}.$$

Symmetry breaking in higher dimensions, 1

How to do computation in higher-dimensional / real cases

- For $X = \mathbb{R}^d$ and $\Gamma \simeq \mathbb{Z}^d$: there is a **cofinal** subset of $\mathfrak{G}$, consisting of subgroups of $\mathbb{Z}^d$ given by

$$\{(mx_1, \dots, mx_d) : (x_1, \dots, x_d) \in \mathbb{Z}^d\}, \quad m \in \mathbb{N}.$$

Cofinality implies that

$$\varinjlim_{G \in \mathfrak{G}} C^*(X)^G \simeq \varinjlim_{G \in m\mathbb{Z}^d} C^*(X)^G.$$

Symmetry breaking in higher dimensions, 1

How to do computation in higher-dimensional / real cases

- For $X = \mathbb{R}^d$ and $\Gamma \simeq \mathbb{Z}^d$: there is a **cofinal** subset of $\mathfrak{G}$, consisting of subgroups of $\mathbb{Z}^d$ given by

$$\{(mx_1, \dots, mx_d) : (x_1, \dots, x_d) \in \mathbb{Z}^d\}, \quad m \in \mathbb{N}.$$

Cofinality implies that

$$\varinjlim_{G \in \mathfrak{G}} C^*(X)^G \simeq \varinjlim_{G \in m\mathbb{Z}^d} C^*(X)^G.$$

- One can employ Künneth Theorem to compute the K-theory of the tensor product

$$C_r^*(m\mathbb{Z}^d) \simeq \bigotimes_{j=1}^d C_r^*(m\mathbb{Z}).$$

Symmetry breaking in higher dimensions, 2

- For KO-theory: use Boersema's real Künneth Theorem:

$$KO_*(C_r^*(m\mathbb{Z}^d)_{\mathbb{R}}) \simeq KO_*(C_r^*(m\mathbb{Z})_{\mathbb{R}}) \otimes_{KO_*(\mathbb{R})} \cdots \otimes_{KO_*(\mathbb{R})} KO_*(C_r^*(m\mathbb{Z})_{\mathbb{R}}).$$

$\underbrace{\hspace{15em}}_{d\text{-copies of } KO_*(C_r^*(m\mathbb{Z})_{\mathbb{R}})}$

Symmetry breaking in higher dimensions, 2

- For KO-theory: use Boersema's real Künneth Theorem:

$$KO_*(C_r^*(m\mathbb{Z}^d)_{\mathbb{R}}) \simeq KO_*(C_r^*(m\mathbb{Z})_{\mathbb{R}}) \otimes_{KO_*(\mathbb{R})} \cdots \otimes_{KO_*(\mathbb{R})} KO_*(C_r^*(m\mathbb{Z})_{\mathbb{R}}).$$

$\underbrace{\hspace{15em}}_{d\text{-copies of } KO_*(C_r^*(m\mathbb{Z})_{\mathbb{R}})}$

Some results of physical relevance

$$K_0(C^*(\mathbb{R}^2)^{\mathfrak{S}}) \simeq \mathbb{Q} \oplus \mathbb{Z} \quad (\text{Chern number})$$

$$KO_4(C^*(\mathbb{R}^2)^{\mathfrak{S}}) \simeq \mathbb{Q} \oplus \mathbb{Z}/2 \quad (\text{Kane–Mele invariant})$$

$$KO_4(C^*(\mathbb{R}^3)^{\mathfrak{S}}) \simeq \mathbb{Q} \oplus \mathbb{Z}/2 \quad (\text{strong } \mathbb{Z}/2\text{-invariant}).$$

Upshot: “stacked” phases are killed by symmetry breaking maps!

Symmetry breaking in higher dimensions, 3

Theorem $K(O)$ -theory of $C^*(\mathbb{R}^d)^G$ for a twisted/real $\mathbb{R}^d$ - $\mathbb{Z}^d$ -module:

$d \backslash i$	0	1	2	3	4	5	6	7	8
K_0	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Z}$	$\mathbb{Q}^4$	$\mathbb{Q}^7 \oplus \mathbb{Z}$	$\mathbb{Q}^{16}$	$\mathbb{Q}^{31} \oplus \mathbb{Z}$	$\mathbb{Q}^{64}$	$\mathbb{Q}^{127} \oplus \mathbb{Z}$
K_1	0	$\mathbb{Z}$	$\mathbb{Q}^2$	$\mathbb{Q}^3 \oplus \mathbb{Z}$	$\mathbb{Q}^8$	$\mathbb{Q}^{15} \oplus \mathbb{Z}$	$\mathbb{Q}^{32}$	$\mathbb{Q}^{63} \oplus \mathbb{Z}$	$\mathbb{Q}^{128}$

$d \backslash i$	0	1	2	3	4	5	6	7	8
KO_0	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Z}$	$\mathbb{Q}^6$	$\mathbb{Q}^{16} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{36} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{71} \oplus \mathbb{Z}$
KO_1	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Q}^2$	$\mathbb{Q}^3$	$\mathbb{Q}^4$	$\mathbb{Q}^5 \oplus \mathbb{Z}$	$\mathbb{Q}^{12}$	$\mathbb{Q}^{28} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{64} \oplus \mathbb{Z}/2$
KO_2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Q}^3$	$\mathbb{Q}^6$	$\mathbb{Q}^{10}$	$\mathbb{Q}^{15} \oplus \mathbb{Z}$	$\mathbb{Q}^{28}$	$\mathbb{Q}^{56} \oplus \mathbb{Z}/2$
KO_3	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Q}^4$	$\mathbb{Q}^{10}$	$\mathbb{Q}^{20}$	$\mathbb{Q}^{35} \oplus \mathbb{Z}$	$\mathbb{Q}^{64}$
KO_4	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Z}/2$	$\mathbb{Q} \oplus \mathbb{Z}/2$	$\mathbb{Q} \oplus \mathbb{Z}$	$\mathbb{Q}^6$	$\mathbb{Q}^{16}$	$\mathbb{Q}^{36}$	$\mathbb{Q}^{70} \oplus \mathbb{Z}$
KO_5	0	$\mathbb{Z}$	$\mathbb{Q}^2$	$\mathbb{Q}^3 \oplus \mathbb{Z}/2$	$\mathbb{Q}^4 \oplus \mathbb{Z}/2$	$\mathbb{Q}^5 \oplus \mathbb{Z}$	$\mathbb{Q}^{12}$	$\mathbb{Q}^{28}$	$\mathbb{Q}^{64}$
KO_6	0	0	$\mathbb{Z}$	$\mathbb{Q}^3$	$\mathbb{Q}^6 \oplus \mathbb{Z}/2$	$\mathbb{Q}^{10} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{15} \oplus \mathbb{Z}$	$\mathbb{Q}^{28}$	$\mathbb{Q}^{56}$
KO_7	0	0	0	$\mathbb{Z}$	$\mathbb{Q}^4$	$\mathbb{Q}^{10} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{20} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{35} \oplus \mathbb{Z}$	$\mathbb{Q}^{64}$

Symmetry breaking in higher dimensions, 3

Theorem $K(O)$ -theory of $C^*(\mathbb{R}^d)^\mathbb{G}$ for a twisted/real $\mathbb{R}^d$ - $\mathbb{Z}^d$ -module:

$d \backslash i$	0	1	2	3	4	5	6	7	8
K_0	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Z}$	$\mathbb{Q}^4$	$\mathbb{Q}^7 \oplus \mathbb{Z}$	$\mathbb{Q}^{16}$	$\mathbb{Q}^{31} \oplus \mathbb{Z}$	$\mathbb{Q}^{64}$	$\mathbb{Q}^{127} \oplus \mathbb{Z}$
K_1	0	$\mathbb{Z}$	$\mathbb{Q}^2$	$\mathbb{Q}^3 \oplus \mathbb{Z}$	$\mathbb{Q}^8$	$\mathbb{Q}^{15} \oplus \mathbb{Z}$	$\mathbb{Q}^{32}$	$\mathbb{Q}^{63} \oplus \mathbb{Z}$	$\mathbb{Q}^{128}$

$d \backslash i$	0	1	2	3	4	5	6	7	8
KO_0	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Z}$	$\mathbb{Q}^6$	$\mathbb{Q}^{16} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{36} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{71} \oplus \mathbb{Z}$
KO_1	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Q}^2$	$\mathbb{Q}^3$	$\mathbb{Q}^4$	$\mathbb{Q}^5 \oplus \mathbb{Z}$	$\mathbb{Q}^{12}$	$\mathbb{Q}^{28} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{64} \oplus \mathbb{Z}/2$
KO_2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Q}^3$	$\mathbb{Q}^6$	$\mathbb{Q}^{10}$	$\mathbb{Q}^{15} \oplus \mathbb{Z}$	$\mathbb{Q}^{28}$	$\mathbb{Q}^{56} \oplus \mathbb{Z}/2$
KO_3	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Q}^4$	$\mathbb{Q}^{10}$	$\mathbb{Q}^{20}$	$\mathbb{Q}^{35} \oplus \mathbb{Z}$	$\mathbb{Q}^{64}$
KO_4	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Z}/2$	$\mathbb{Q} \oplus \mathbb{Z}/2$	$\mathbb{Q} \oplus \mathbb{Z}$	$\mathbb{Q}^6$	$\mathbb{Q}^{16}$	$\mathbb{Q}^{36}$	$\mathbb{Q}^{70} \oplus \mathbb{Z}$
KO_5	0	$\mathbb{Z}$	$\mathbb{Q}^2$	$\mathbb{Q}^3 \oplus \mathbb{Z}/2$	$\mathbb{Q}^4 \oplus \mathbb{Z}/2$	$\mathbb{Q}^5 \oplus \mathbb{Z}$	$\mathbb{Q}^{12}$	$\mathbb{Q}^{28}$	$\mathbb{Q}^{64}$
KO_6	0	0	$\mathbb{Z}$	$\mathbb{Q}^3$	$\mathbb{Q}^6 \oplus \mathbb{Z}/2$	$\mathbb{Q}^{10} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{15} \oplus \mathbb{Z}$	$\mathbb{Q}^{28}$	$\mathbb{Q}^{56}$
KO_7	0	0	0	$\mathbb{Z}$	$\mathbb{Q}^4$	$\mathbb{Q}^{10} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{20} \oplus \mathbb{Z}/2$	$\mathbb{Q}^{35} \oplus \mathbb{Z}$	$\mathbb{Q}^{64}$

A dichotomy for weak/strong K-theory, 1

The **reduced** part of $K(O)$ -theory of $C^*(\mathbb{R}^d)^G$:

$i \backslash d$	0	1	2	3	4	5	6	7	8
sK_0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
sK_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

$i \backslash d$	0	1	2	3	4	5	6	7	8
sKO_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
sKO_1	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
sKO_2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$
sKO_3	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
sKO_4	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
sKO_5	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
sKO_6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
sKO_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0

This is the same as Kitaev's periodic table / homotopy groups of $U(\infty)$ or $O(\infty)$ / $K(O)$ -theory of Roe algebra.

A dichotomy for weak/strong K-theory, 2

$K_i(C^*(X)^{\mathfrak{S}})$ decomposes as a (non-canonical) direct sum of a **maximal divisible** subgroup and a **reduced** quotient:

$$0 \rightarrow \underbrace{wK_*(C^*(X)^{\mathfrak{S}})}_{\text{weak colimit}} \rightarrow K_*(C^*(X)^{\mathfrak{S}}) \rightarrow \underbrace{sk_*(C^*(X)^{\mathfrak{S}})}_{\text{strong quotient}} \rightarrow 0.$$

A dichotomy for weak/strong K-theory, 2

$K_i(C^*(X)^{\mathfrak{G}})$ decomposes as a (non-canonical) direct sum of a **maximal divisible** subgroup and a **reduced** quotient:

$$0 \rightarrow \underbrace{wK_*(C^*(X)^{\mathfrak{G}})}_{\text{weak colimit}} \rightarrow K_*(C^*(X)^{\mathfrak{G}}) \rightarrow \underbrace{sk_*(C^*(X)^{\mathfrak{G}})}_{\text{strong quotient}} \rightarrow 0.$$

Definition For each $G \in \mathfrak{G}$, denote by $\iota_{\mathfrak{G} \setminus G}: C^*(X)^G \hookrightarrow C^*(X)^{\mathfrak{G}}$ the universal map into the colimit.

A dichotomy for weak/strong K-theory, 2

$K_i(C^*(X)^\mathfrak{G})$ decomposes as a (non-canonical) direct sum of a **maximal divisible** subgroup and a **reduced** quotient:

$$0 \rightarrow \underbrace{wK_*(C^*(X)^\mathfrak{G})}_{\text{weak colimit}} \rightarrow K_*(C^*(X)^\mathfrak{G}) \rightarrow \underbrace{sk_*(C^*(X)^\mathfrak{G})}_{\text{strong quotient}} \rightarrow 0.$$

Definition For each $G \in \mathfrak{G}$, denote by $\iota_{\mathfrak{G} \setminus G} : C^*(X)^G \hookrightarrow C^*(X)^\mathfrak{G}$ the universal map into the colimit.

- The **weak subgroup**

$$wK_i(C^*(X)^G) := K_i(\iota_{\mathfrak{G} \setminus G})^{-1}(wK_i(C^*(X)^\mathfrak{G}))$$

- The **strong quotient**

$$sk_i(C^*(X)^G) := K_i(C^*(X)^G) / wK_i(C^*(X)^G)$$

Universal property of strong K-theory, 1

- For every abelian group A , let W be its maximal divisible subgroup. Then there is a short exact sequence which splits non-canonically:

$$0 \rightarrow W \xrightarrow{i} A \xrightarrow[\substack{q \\ \downarrow \dots}]{q} A/W \rightarrow 0.$$

Universal property of strong K-theory, 1

- For every abelian group A , let W be its **maximal divisible subgroup**. Then there is a short exact sequence which splits non-canonically:

$$0 \rightarrow W \xrightarrow{i} A \xrightarrow[q, \dots]{q} A/W \rightarrow 0.$$

- Let R be a **reduced** (i.e. possesses a trivial maximal divisible subgroup) abelian group. Then any group homomorphism $A \rightarrow R$ factors:

$$\begin{array}{ccc} A & \xrightarrow{\quad} & R \\ & \searrow q & \nearrow \text{dotted} \\ & A/W & \end{array}$$

Universal property of strong K-theory, 1

- For every abelian group A , let W be its **maximal divisible subgroup**. Then there is a short exact sequence which splits non-canonically:

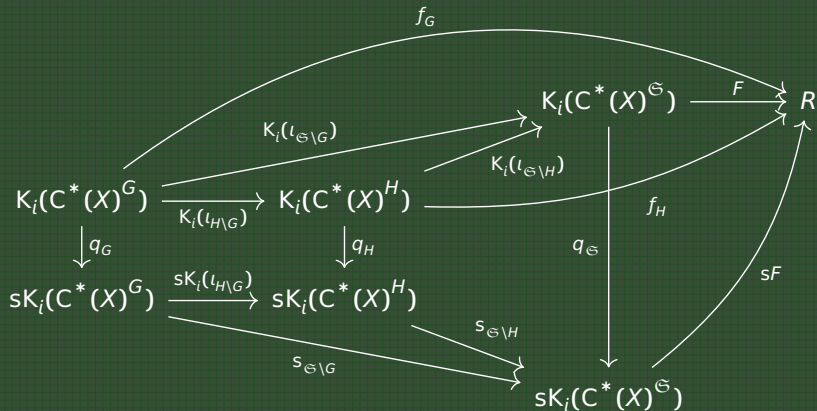
$$0 \rightarrow W \xrightarrow{i} A \xrightarrow[\text{f...}]{q} A/W \rightarrow 0.$$

- Let R be a **reduced** (i.e. possesses a trivial maximal divisible subgroup) abelian group. Then any group homomorphism $A \rightarrow R$ factors:

$$\begin{array}{ccc} A & \xrightarrow{\quad} & R \\ & \searrow q & \nearrow \text{dotted} \\ & A/W & \end{array}$$

- Assume **consistency with symmetry breaking**:
If R is a receptacle of **quantised** invariants, then $K_i(C^*(X)^G) \rightarrow R$ must factor through $sK_i(C^*(X)^G) \rightarrow R$.

Universal property of strong K-theory, 2



Strong-to-coarse comparison maps, 1

- A useful receptacle reduced abelian group R :
K(O)-theory of **non-equivariant Roe algebras**.

$$K_*(C^*(\mathbb{R}^d)) \simeq K_{*-d}(\mathbb{C}), \quad KO_*(C^*(\mathbb{R}^d)) \simeq KO_{*-d}(\mathbb{R}).$$

Strong-to-coarse comparison maps, 1

- A useful receptacle reduced abelian group R : $K(O)$ -theory of **non-equivariant Roe algebras**.

$$K_*(C^*(\mathbb{R}^d)) \simeq K_{*-d}(\mathbb{C}), \quad KO_*(C^*(\mathbb{R}^d)) \simeq KO_{*-d}(\mathbb{R}).$$

- Forgetting the equivariance altogether gives the “strong-to-coarse comparison map” for each $G \in \mathfrak{S}$:

$$sK_i(\iota_{1 \setminus G}): sK_i(C^*(X, \mathcal{H})^G) \rightarrow K_i(C^*(X, \mathcal{H}))$$

Strong-to-coarse comparison maps, 1

- ▶ A useful receptacle reduced abelian group R : $K(O)$ -theory of **non-equivariant Roe algebras**.

$$K_*(C^*(\mathbb{R}^d)) \simeq K_{*-d}(\mathbb{C}), \quad KO_*(C^*(\mathbb{R}^d)) \simeq KO_{*-d}(\mathbb{R}).$$

- ▶ Forgetting the equivariance altogether gives the “strong-to-coarse comparison map” for each $G \in \mathfrak{S}$:

$$sK_i(\iota_{1 \setminus G}): sK_i(C^*(X, \mathcal{H})^G) \rightarrow K_i(C^*(X, \mathcal{H}))$$

- ▶ The map $sK_i(\iota_{1 \setminus G})$ can be used to detect non-trivial elements in the strong K -theory of $C^*(X, \mathcal{H})^G$ (equivalently, $C_r^*(G)$).

Strong-to-coarse comparison maps, 2

Theorem Let

Γ finitely generated, torsion free, nilpotent discrete group.

$\mathcal{H}$ ample, σ -twisted complex X - Γ -module
satisfying the basic setup.

σ magnetic 2-cocycle for $(\Gamma_{\mathbb{R}}, \Gamma)$
($\Gamma_{\mathbb{R}}$: Mal'cev completion of Γ).

Then for every $G \in \mathfrak{S}$, the strong-to-coarse comparison maps

$$sK_i(C^*(X, \mathcal{H})^G) \rightarrow K_i(C^*(X, \mathcal{H}))$$

are surjective.

Strong-to-coarse comparison maps, 2

Theorem Let

Γ finitely generated, torsion free, nilpotent discrete group.

$\mathcal{H}$ ample, σ -twisted complex X - Γ -module
satisfying the basic setup.

σ magnetic 2-cocycle for $(\Gamma_{\mathbb{R}}, \Gamma)$
($\Gamma_{\mathbb{R}}$: Mal'cev completion of Γ).

Then for every $G \in \mathfrak{S}$, the strong-to-coarse comparison maps

$$sK_i(C^*(X, \mathcal{H})^G) \rightarrow K_i(C^*(X, \mathcal{H}))$$

are surjective.

Proof. The generators of $K_i(C^*(\Gamma_{\mathbb{R}}, \mathcal{H}))$ is the (coarse) index class of the spin Dirac operator on $\Gamma_{\mathbb{R}}$. It is G -invariant for every G (using contractibility of $\Gamma_{\mathbb{R}}$). So it lifts to every $sK_i(C^*(\Gamma_{\mathbb{R}}, \mathcal{H})^G)$. □

Application: TKNN–Chern number of Landau levels

- Consider the **twisted** Dirac operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

$$\mathcal{D}_b = \begin{pmatrix} 0 & \mathcal{D}_b^- \\ \mathcal{D}_b^+ & 0 \end{pmatrix} = -2i \begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix}.$$

Application: TKNN–Chern number of Landau levels

- Consider the **twisted** Dirac operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

$$\mathcal{D}_b = \begin{pmatrix} 0 & \mathcal{D}_b^- \\ \mathcal{D}_b^+ & 0 \end{pmatrix} = -2i \begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix}.$$

- **Lowest Landau level** $:= \ker \mathcal{D}_b^+ = \overline{\text{span}}_{m \in \mathbb{N}} \left\{ z \mapsto z^m e^{-b|z|^2/4} \right\}.$

Application: TKNN–Chern number of Landau levels

- Consider the **twisted** Dirac operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

$$\mathcal{D}_b = \begin{pmatrix} 0 & \mathcal{D}_b^- \\ \mathcal{D}_b^+ & 0 \end{pmatrix} = -2i \begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix}.$$

- **Lowest Landau level** $:= \ker \mathcal{D}_b^+ = \overline{\text{span}}_{m \in \mathbb{N}} \left\{ z \mapsto z^m e^{-b|z|^2/4} \right\}$.
- The eigenprojection onto LLL

$$[P_{\text{LLL}, b}] \in K_0(C^*(\mathbb{R}^2))$$

is a generator (deep result by John Roe).

Application: TKNN–Chern number of Landau levels

- Consider the **twisted** Dirac operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

$$\mathcal{D}_b = \begin{pmatrix} 0 & \mathcal{D}_b^- \\ \mathcal{D}_b^+ & 0 \end{pmatrix} = -2i \begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix}.$$

- **Lowest Landau level** $:= \ker \mathcal{D}_b^+ = \overline{\text{span}}_{m \in \mathbb{N}} \left\{ z \mapsto z^m e^{-b|z|^2/4} \right\}.$
- The eigenprojection onto LLL

$$[P_{\text{LLL},b}] \in K_0(C^*(\mathbb{R}^2))$$

is a generator (deep result by John Roe).

- This forces the (TKNN–)Chern number of $P_{\text{LLL},b}$ to be ± 1 :

$$\underbrace{K_0(C^*(X)^G)}_{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{q_G} \underbrace{SK_0(C^*(X)^G)}_{\mathbb{Z}} \xrightarrow[\sim]{SK_0(\iota_{1 \setminus G})} \underbrace{K_0(C^*(X))}_{\mathbb{Z}}$$

$$[P_{\text{LLL},b}]_G = (r, c) \longmapsto c \longmapsto [P_{\text{LLL},b}] = \pm 1.$$

Strong equals coarse?

Question Is every strong-to-coarse comparison map an isomorphism?

More-or-less: let $|\Gamma| := (\Gamma, \text{proper left-invariant metric})$.

Are the induced K-theory maps

$$K_i(C_r^*(\Gamma)) \rightarrow K_i(C^*(|\Gamma|))$$

always surjective?

Strong equals coarse?

Question Is every strong-to-coarse comparison map an isomorphism?

More-or-less: let $|\Gamma| := (\Gamma, \text{proper left-invariant metric})$.

Are the induced K-theory maps

$$K_i(C_r^*(\Gamma)) \rightarrow K_i(C^*(|\Gamma|))$$

always surjective?

Idea Counterexamples may arise because:

- ▶ $K_i(C^*(|\Gamma|))$ is often singly generated by the “coarse-geometric fundamental class”.

Strong equals coarse?

Question Is every strong-to-coarse comparison map an isomorphism?

More-or-less: let $|\Gamma| := (\Gamma, \text{proper left-invariant metric})$.

Are the induced K-theory maps

$$K_i(C_r^*(\Gamma)) \rightarrow K_i(C^*(|\Gamma|))$$

always surjective?

Idea Counterexamples may arise because:

- ▶ $K_i(C^*(|\Gamma|))$ is often singly generated by the “coarse-geometric fundamental class”.
- ▶ It lifts to the “(coarse) K-homological fundamental class” under the coarse Baum–Connes assembly map.

Strong equals coarse?

Question Is every strong-to-coarse comparison map an isomorphism?

More-or-less: let $|\Gamma| := (\Gamma, \text{proper left-invariant metric})$.

Are the induced K-theory maps

$$K_i(C_r^*(\Gamma)) \rightarrow K_i(C^*(|\Gamma|))$$

always surjective?

Idea Counterexamples may arise because:

- ▶ $K_i(C^*(|\Gamma|))$ is often singly generated by the “coarse-geometric fundamental class”.
- ▶ It lifts to the “(coarse) K-homological fundamental class” under the coarse Baum–Connes assembly map.
- ▶ Such a class may not come from $K_i(C_r^*(\Gamma))$.

A non-surjective example, 1

Example Let $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$ be the wallpaper group, i.e. the crystallographic group given by

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \rtimes \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \longrightarrow 1 \\ & & (a, b) & \longmapsto & (a, 2b) & & \\ & & & & (a, b) & \longmapsto & b \bmod 2\mathbb{Z}. \end{array}$$

Then:

- $B\Gamma = \text{Klein bottle } K$.

A non-surjective example, 1

Example Let $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$ be the wallpaper group, i.e. the crystallographic group given by

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \rtimes \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \longrightarrow 1 \\ & & (a, b) & \longmapsto & (a, 2b) & & \\ & & & & (a, b) & \longmapsto & b \bmod 2\mathbb{Z}. \end{array}$$

Then:

- ▶ $B\Gamma = \text{Klein bottle } K$.
- ▶ The strong-to-coarse comparison maps

$$sK_i(C^*(K)^\Gamma) \rightarrow K_i(C^*(K))$$

vanishes in all degrees, whereas $K_i(C^*(K)) \simeq K_i(\mathbb{C})$.

A non-surjective example, 1

Example Let $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$ be the wallpaper group, i.e. the crystallographic group given by

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \rtimes \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \longrightarrow 1 \\ & & (a, b) & \longmapsto & (a, 2b) & & \\ & & & & (a, b) & \longmapsto & b \bmod 2\mathbb{Z}. \end{array}$$

Then:

- ▶ $B\Gamma = \text{Klein bottle } K$.
- ▶ The strong-to-coarse comparison maps

$$sK_i(C^*(K)^\Gamma) \rightarrow K_i(C^*(K))$$

vanishes in all degrees, whereas $K_i(C^*(K)) \simeq K_i(\mathbb{C})$.

- ▶ Proof by PV sequence.

A non-surjective example, 2

Notice that $\mathbb{Z} \rtimes \mathbb{Z}$ contains an abelian subgroup isomorphic to $\mathbb{Z}^2$. The strong-to-coarse comparison maps become isomorphisms there.

Theorem Let

Γ finitely generated, torsion free, nilpotent discrete group.

$\tilde{\Gamma}$ extension of a finite group F by Γ .

$\tilde{\mathcal{G}}$ poset of finite-index subgroups of $\tilde{\Gamma}$.

$\mathcal{H}$ ample, σ -twisted complex X - $\tilde{\Gamma}$ -module.

σ 2-cocycle for $\tilde{\Gamma}$,
which restricts to a magnetic 2-cocycle for $(\Gamma_{\mathbb{R}}, \Gamma)$.

A non-surjective example, 2

Notice that $\mathbb{Z} \rtimes \mathbb{Z}$ contains an abelian subgroup isomorphic to $\mathbb{Z}^2$. The strong-to-coarse comparison maps become isomorphisms there.

Theorem Let

Γ finitely generated, torsion free, nilpotent discrete group.

$\tilde{\Gamma}$ extension of a finite group F by Γ .

$\tilde{\mathfrak{S}}$ poset of finite-index subgroups of $\tilde{\Gamma}$.

$\mathcal{H}$ ample, σ -twisted complex X - $\tilde{\Gamma}$ -module.

σ 2-cocycle for $\tilde{\Gamma}$,

which restricts to a magnetic 2-cocycle for $(\Gamma_{\mathbb{R}}, \Gamma)$.

Then the strong-colimit-to-coarse comparison maps

$$sK_i(C^*(X, \mathcal{H})^{\tilde{\mathfrak{S}}}) \rightarrow K_i(C^*(X, \mathcal{H}))$$

are surjective.

Summary and outlook

- Instead of looking at $K_*(C^*(X)^\Gamma)$, it may be useful to look at the whole $\mathfrak{G}$ -shaped diagram

$$\{K_*(C^*(X)^G) \mid [\Gamma : G] < +\infty\}.$$

Summary and outlook

- Instead of looking at $K_*(C^*(X)^\Gamma)$, it may be useful to look at the whole $\mathfrak{G}$ -shaped diagram

$$\{K_*(C^*(X)^G) \mid [\Gamma : G] < +\infty\}.$$

This allows for a dichotomy of “strong/weak” K-theory elements by looking at the **divisibility** of their images in the colimit.

Summary and outlook

- Instead of looking at $K_*(C^*(X)^\Gamma)$, it may be useful to look at the whole $\mathfrak{G}$ -shaped diagram

$$\{K_*(C^*(X)^G) \mid [\Gamma : G] < +\infty\}.$$

This allows for a dichotomy of “strong/weak” K-theory elements by looking at the **divisibility** of their images in the colimit.

- In many cases (appearing naturally in physics!):
“strong” equals “coarse”,
but goes wrong already for crystallographic groups.

Summary and outlook

- Instead of looking at $K_*(C^*(X)^\Gamma)$, it may be useful to look at the whole $\mathfrak{S}$ -shaped diagram

$$\{K_*(C^*(X)^G) \mid [\Gamma : G] < +\infty\}.$$

This allows for a dichotomy of “strong/weak” K-theory elements by looking at the **divisibility** of their images in the colimit.

- In many cases (appearing naturally in physics!):
 “strong” equals “coarse”,
but goes wrong already for crystallographic groups.
- More generally: instead of looking at all finite-index subgroups, it is one may look at smaller subposets of $\mathfrak{S}$.

Summary and outlook

- Instead of looking at $K_*(C^*(X)^\Gamma)$, it may be useful to look at the whole $\mathfrak{S}$ -shaped diagram

$$\{K_*(C^*(X)^G) \mid [\Gamma : G] < +\infty\}.$$

This allows for a dichotomy of “strong/weak” K-theory elements by looking at the **divisibility** of their images in the colimit.

- In many cases (appearing naturally in physics!):

“strong” equals “coarse”,

but goes wrong already for crystallographic groups.

- More generally: instead of looking at all finite-index subgroups, it is one may look at smaller subposets of $\mathfrak{S}$.

This allows for maintaining the presence of certain point groups.

Summary and outlook

- Instead of looking at $K_*(C^*(X)^\Gamma)$, it may be useful to look at the whole $\mathfrak{S}$ -shaped diagram

$$\{K_*(C^*(X)^G) \mid [\Gamma : G] < +\infty\}.$$

This allows for a dichotomy of “strong/weak” K-theory elements by looking at the **divisibility** of their images in the colimit.

- In many cases (appearing naturally in physics!):

“strong” equals “coarse”,

but goes wrong already for crystallographic groups.

- More generally: instead of looking at all finite-index subgroups, it is one may look at smaller subposets of $\mathfrak{S}$.

This allows for maintaining the presence of certain point groups.

Thanks for your attention!

Appendix: Magnetic twists, 1

Let X be a connected and simply-connected manifold with an action of a group Γ by diffeomorphisms, together with

1. A **trivializable** Hermitian line bundle $\mathcal{L} \rightarrow X$;
2. A connection $\nabla = d - i\mathcal{A}$ on $\mathcal{L}$ with Γ -invariant curvature.

Appendix: Magnetic twists, 1

Let X be a connected and simply-connected manifold with an action of a group Γ by diffeomorphisms, together with

1. A **trivializable** Hermitian line bundle $\mathcal{L} \rightarrow X$;
2. A connection $\nabla = d - i\mathcal{A}$ on $\mathcal{L}$ with Γ -invariant curvature.

Choose an origin $\mathcal{O} \in X$.

► The phase factor

$$I_g(x) := I_g^{(\mathcal{A})}(x) = \int_{\mathcal{O}}^x (\mathcal{A} - g^* \mathcal{A}), \quad g \in \Gamma, x \in X,$$

► The **magnetic translation**

$$U_g := U_g^{(\mathcal{A})} = e^{iI_g^{-1}} \cdot (g^{-1})^*, \quad g \in \Gamma.$$

Appendix: Magnetic twists, 2

- These magnetic translation operators generate a PUR of Γ :

$$U_g^{(\mathcal{A})} U_h^{(\mathcal{A})} = \sigma(g, h) U_{gh}^{(\mathcal{A})}, \quad g, h \in \Gamma,$$

where $\sigma(g^{-1}, h^{-1})$ is defined as

$$\begin{aligned} & \exp\left(i(I_g + g^* I_h - I_{hg})\right) \\ &= \exp\left(\int_{\mathcal{O}}^x (\mathcal{A} - g^* \mathcal{A}) + \int_{\mathcal{O}}^{g \cdot x} (\mathcal{A} - h^* \mathcal{A}) - \int_{\mathcal{O}}^x (\mathcal{A} - (hg)^* \mathcal{A})\right) \\ &= \exp\left(i \int_{\mathcal{O}}^{g \cdot \mathcal{O}} (\mathcal{A} - h^* \mathcal{A})\right) \end{aligned}$$

Appendix: Magnetic twists, 2

- These magnetic translation operators generate a PUR of Γ :

$$U_g^{(\mathcal{A})} U_h^{(\mathcal{A})} = \sigma(g, h) U_{gh}^{(\mathcal{A})}, \quad g, h \in \Gamma,$$

where $\sigma(g^{-1}, h^{-1})$ is defined as

$$\begin{aligned} & \exp\left(i(I_g + g^* I_h - I_{hg})\right) \\ &= \exp\left(\int_{\mathcal{O}}^x (\mathcal{A} - g^* \mathcal{A}) + \int_{\mathcal{O}}^{g \cdot x} (\mathcal{A} - h^* \mathcal{A}) - \int_{\mathcal{O}}^x (\mathcal{A} - (hg)^* \mathcal{A})\right) \\ &= \exp\left(i \int_{\mathcal{O}}^{g \cdot \mathcal{O}} (\mathcal{A} - h^* \mathcal{A})\right) \end{aligned}$$

Definition A 2-cocycle $\omega: \Gamma \times \Gamma \rightarrow \text{U}(1)$ is said to be a **magnetic 2-cocycle for the pair (X, Γ)** , if there exists a Hermitian line bundle $\mathcal{L} \rightarrow X$ with connection having Γ -invariant curvature, such that the 2-cocycle σ as above is cohomologous to ω .

Go back