Models of aperiodic topological insulators

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Topological insulators

- Topological insulators are materials that are insulating in the bulk but allow a current to flow on the boundary.
- The current flowing on the current is usually quite robust under disorder. This illustrates the adjective topological.
- "Classically", periodic topological insulators on a lattice ($\simeq \mathbb{Z}^d$) are modelled by vector bundles over a torus $\mathbb{T}^d$ called the Brillouin zone. The $\mathbb{T}^d$ is the character space of the lattice from Fourier transform.
- Topological insulators are classified by the K-theory of the Brillouin zone.

Aperiodicity

On a periodic lattice:

- The magnetic field destroys the periodicity, turning the Brillouin zone into a NC torus.
- Impurity also destroys the translation invariance (periodicity) of the Hamiltonian.
- Integer quantum Hall effect: the impurity is essential for Anderson localisation.

Moreover:

- Physicists have constructed amorphous topological insulators modelled on random point patterns.
- No useful $\mathbb{Z}^d$-labelling for such point patterns. $\Longrightarrow$ Generic aperiodicity
- We model such a point pattern by a Delone set $\mathfrak{L}$.

Observable algebras

- The noncommutative geometry or dynamics of a topological insulator is described by a C^*-algebra called the observable algebra.
  In the "commutative" case, it is just $\mathrm{C}(\mathbb{T}^d)$ (up to stablisation by compact operators).
- It should be "large" enough to contain all possible Hamiltonians, and "small" enough to have useful homotopy theory (K-theory).
  "Minimal" choice Groupoid C^*-algebra of Bellissard and Kellendonk.
  "Maximal" choice Roe C^*-algebra used by Ewert and Meyer.
- These different models also fit in the "universality" of dynamical systems.
- I will say something on the index theory (K-theory) level.

Groupoid models

Bellissard, Kellondonk, Prodan, Bourne, Mesland, ...

- Let $0 < r < R$. A discrete infinite set $\mathfrak{L} \subseteq \mathbb{R}^d$ is called an (r, R)-Delone set if $|\mathrm{B}(x, r) \cap \mathfrak{L}| \leq 1$ and $|\mathrm{B}(x, R) \cap \mathfrak{L}| \geq 1$ for all $x \in \mathbb{R}^d$.
- The set of (r, R)-Delone set $\mathrm{Del}_{(r,R)}(\mathbb{R}^d)$ is a compact metric space which carries a continuous action of $\mathbb{R}^d$ by translation.
- So there is a topological dynamical system $(\mathrm{Del}_{(r,R)}(\mathbb{R}^d) \curvearrowleft \mathbb{R}^d)$, or $(\Omega_{\mathfrak{L}} \curvearrowleft \mathbb{R}^d)$ where $\Omega_{\mathfrak{L}}$ is the orbit of $\mathfrak{L}$.
- The dynamical system generates the action groupoid $\Omega_{\mathfrak{L}} \rtimes \mathbb{R}^d \rightrightarrows \Omega_{\mathfrak{L}}$. Taking its transversal gives an étale groupoid $\mathcal{G}_{\mathfrak{L}} \rightrightarrows \Omega_0$.
- The groupoid C^*-algebra $\mathrm{C}^*(\mathcal{G}_{\mathfrak{L}})$ contains all the "translates" of the Hamiltonian witnessed on a single site $\omega \in \mathfrak{L}$.

Coarse-geometric models

Kubota, Ewert–Meyer, Ludewig–Thiang, ...

- The discrete metric space $\mathfrak{L}$ yields also coarse-geometric C^*-algebras.
- Kubota uses the uniform Roe C^*-algebra $\mathrm{C}^*_{\mathrm{u,Roe}}(\mathfrak{L})$ as the observable algebra. It contains operators which have finite propagation.
- Ewert and Meyer choose the Roe C^*-algebra $\mathrm{C}^*_{\mathrm{Roe}}(\mathfrak{L})$ of locally compact, finite propagation operators. The advantage is that it has simpler K-theory and does not require a fixed number of valence bands.
- There is an inclusion

  $$\iota_\omega \colon \mathrm{C}^*(\mathcal{G}_{\mathfrak{L}}) \hookrightarrow \mathrm{C}^*_{\mathrm{Roe}}(\mathfrak{L})$$

  which depends on an element $\omega \in \mathfrak{L}$. The choice is due to the representation of $\mathrm{C}^*_{\mathrm{Roe}}(\mathfrak{L})$.

Index computation

- Physical observable quantities ($\#$ edge states, electrical conductance on the edge, …) are index pairings between the K-theory class of an insulator and a "fundamental class" in K-homology.
- In the Roe C^*-algebra case, the fundamental class is given by the following d-summable spectral triple (c.f. Kasparov, Ewert–Meyer):

$$\mathcal{E}_{\mathrm{Roe}} = \left(\mathbb{C}_{\mathrm{Roe}}[\mathfrak{L}] \hat{\otimes} \mathrm{Cl}_{0,d}, \quad \ell^2(\mathfrak{L}, \mathscr{K}) \hat{\otimes} \bigwedge^* \mathbb{R}^d, \quad \sum_{j=1}^{d} X_j \hat{\otimes} \gamma_j \right)$$

- For the Kellondonk–Bellissard groupoid C^*-algebra, the fundamental class is given by the composition of a "bulk cycle" (an unbounded Kasparov $(\mathrm{C}^*(\mathcal{G}_{\mathfrak{L}}), \mathrm{C}(\Omega_0))$-module) and an evaluation map (c.f. Bourne–Mesland):

$$\mathcal{E}_{\mathcal{G}} = \left(\mathrm{C}_c(\mathcal{G}_{\mathfrak{L}}) \hat{\otimes} \mathrm{Cl}_{0,d}, \quad E_{\mathrm{C}(\Omega_0)} \hat{\otimes} \bigwedge^* \mathbb{R}^d, \quad \sum_{i=1}^{d} X_j \hat{\otimes} \gamma_j \right),$$

$$\mathrm{ev}_\omega \colon \mathrm{C}(\Omega_0) \to \mathbb{C}, \quad f \mapsto f(\mathfrak{L} + \omega).$$

Index computation, cont.

A factorisation:

Theorem

The following diagram commutes in the Kasparov category:

$$\begin{tikzcd} C^*(\mathcal{G}_{\mathfrak{L}}) \arrow[r, "\mathcal{E}_{\mathcal{G}}"] \arrow[d, hook, "\iota_\omega"'] & C(\Omega_0) \arrow[d, "\mathrm{ev}_\omega"] \\ C^*_{\mathrm{Roe}}(\mathfrak{L}) \arrow[r, "\mathcal{E}_{\mathrm{Roe}}"'] & \mathbb{C}, \end{tikzcd}$$

Ongoing work:

- This factorisation in the mobility gap regime (Anderson localisation).
- The bulk–edge correspondence.
- The interacting case (operator algebraic approach to topological order).