Models of aperiodic topological insulators



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Mathematical institute, Leiden university Göttingen, October 10, 2023



Topological insulators

- Topological insulators are materials that are insulating in the bulk but allow a current to flow on the boundary.
- The current flowing on the current is usually quite robust under disorder. This illustrates the adjective topological.
- Classically", periodic topological insulators on a lattice (≃ Z^d) are modelled by vector bundles over a torus T^d called the Brillouin zone. The T^d is the character space of the lattice from Fourier transform.
- Topological insulators are classified by the K-theory of the Brillouin zone.

Aperiodicity

On a periodic lattice:

- The magnetic field destroys the periodicity, turning the Brillouin zone into a NC torus.
- Impurity also destroys the translation invariance (periodicity) of the Hamiltonian.
- Integer quantum Hall effect: the impurity is essential for Anderson localisation.

Moreover:

- Physicists have constructed amorphous topological insulators modelled on random point patterns.
- ▶ No useful \mathbb{Z}^d -labelling for such point patterns. \implies Generic aperiodicity
- We model such a point pattern by a Delone set \mathfrak{L} .

Observable algebras

The noncommutative geometry or dynamics of a topological insulator is described by a C*-algebra called the observable algebra. In the "commutative" case, it is just C(T^d) (up to stablisation by compact operators).

 It should be "large" enough to contain all possible Hamiltonians, and "small" enough to have useful homotopy theory (K-theory).
"Minimal" choice Groupoid C*-algebra of Bellissard and Kellendonk.
"Maximal" choice Roe C*-algebra used by Ewert and Meyer.

- These different models also fit in the "universality" of dynamical systems.
- I will say something on the index theory (K-theory) level.

Groupoid models

Bellissard, Kellondonk, Prodan, Bourne, Mesland, ...

- ▶ Let 0 < r < R. A discrete infinite set $\mathfrak{L} \subseteq \mathbb{R}^d$ is called an (r, R)-Delone set if $|B(x, r) \cap \mathfrak{L}| \le 1$ and $|B(x, R) \cap \mathfrak{L}| \ge 1$ for all $x \in \mathbb{R}^d$.
- ▶ The set of (r, R)-Delone set $\text{Del}_{(r,R)}(\mathbb{R}^d)$ is a compact metric space which carries a continuous action of \mathbb{R}^d by translation.
- So there is a topological dynamical system $(\mathrm{Del}_{(r,R)}(\mathbb{R}^d) \curvearrowleft \mathbb{R}^d)$, or $(\Omega_{\mathfrak{L}} \curvearrowleft \mathbb{R}^d)$ where $\Omega_{\mathfrak{L}}$ is the orbit of \mathfrak{L} .
- The dynamical system generates the action groupoid Ω_L ⋊ ℝ^d ⇒ Ω_L. Taking its transversal gives an étale groupoid G_L ⇒ Ω₀.
- ▶ The groupoid C*-algebra C*($\mathcal{G}_{\mathfrak{L}}$) contains all the "translates" of the Hamiltonian witnessed on a single site $\omega \in \mathfrak{L}$.

Coarse-geometric models

Kubota, Ewert-Meyer, Ludewig-Thiang, ...

- ▶ The discrete metric space \mathfrak{L} yields also coarse-geometric C^* -algebras.
- ▶ Kubota uses the uniform Roe C*-algebra C^{*}_{u,Roe}(£) as the observable algebra. It contains operators which have finite propagation.
- ▶ Ewert and Meyer choose the Roe C*-algebra C^{*}_{Roe}(£) of locally compact, finite propagation operators. The advantage is that it has simpler K-theory and does not require a fixed number of valence bands.
- There is an inclusion

$$\iota_{\omega}\colon \mathrm{C}^*(\mathcal{G}_{\mathfrak{L}})\hookrightarrow \mathrm{C}^*_{\mathrm{Roe}}(\mathfrak{L})$$

which depends on an element $\omega \in \mathfrak{L}$. The choice is due to the representation of $C^*_{\mathrm{Roe}}(\mathfrak{L})$.

Index computation

- Physical observable quantities (# edge states, electrical conductance on the edge, ...) are index pairings between the K-theory class of an insulator and a "fundamental class" in K-homology.
- In the Roe C*-algebra case, the fundamental class is given by the following d-summable spectral triple (c.f. Kasparov, Ewert–Meyer):

$$\mathcal{E}_{\text{Roe}} = \left(\mathbb{C}_{\text{Roe}}[\mathfrak{L}] \, \hat{\otimes} \, \text{Cl}_{0,d}, \quad \ell^2(\mathfrak{L}, \mathscr{K}) \, \hat{\otimes} \, \bigwedge^* \mathbb{R}^d, \quad \sum_{j=1}^d X_j \, \hat{\otimes} \, \gamma_j \right)$$

For the Kellondonk–Bellissard groupoid C*-algebra, the fundamental class is given by the composition of a "bulk cycle" (an unbounded Kasparov (C*(G_𝔅), C(Ω₀))-module) and an evaluation map (c.f. Bourne–Mesland):

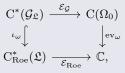
$$\mathcal{E}_{\mathcal{G}} = \left(\operatorname{C}_{c}(\mathcal{G}_{\mathfrak{L}}) \hat{\otimes} \operatorname{Cl}_{0,d}, \quad E_{\operatorname{C}(\Omega_{0})} \hat{\otimes} \bigwedge^{*} \mathbb{R}^{d}, \quad \sum_{i=1}^{d} X_{j} \hat{\otimes} \gamma_{j} \right),$$
$$\operatorname{ev}_{\omega} \colon \operatorname{C}(\Omega_{0}) \to \mathbb{C}, \quad f \mapsto f(\mathfrak{L} + \omega).$$

Index computation, cont.

A factorisation:

Theorem

The following diagram commutes in the Kasparov category:



Ongoing work:

- This factorisation in the mobility gap regime (Anderson localisation).
- ► The bulk-edge correspondence.
- ▶ The interacting case (operator algebraic approach to topological order).