Lecture notes on unbounded KK-theory

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Contents

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Introduction

These notes originate from a course on unbounded KK-theory that I gave at the Hausdorff school "Noncommutative Geometry and Operator Algebras" in Bonn in May of 2023. The material presented here covers the material of that course, with some further examples and detailed arguments added, as well as a comprehensive reference overview of the developments in the area of unbounded KK-theory in the last 15 years.

The viewpoint taken here is that of Connes' noncommutative differential geometry. Unbounded KK-theory serves as organising principle, capturing both the fine geometric as well the more rigid topological aspects of the theory. Although there are still several open challenges and our current understanding is far from complete, the picture is sufficiently clear to make a set of notes like these timely.

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Notation and conventions Let X be a Hilbert B-module. We write $\mathrm{End}^*_B(X)$ or $\mathrm{End}^*(X)$ for the C*-algebra of adjointable operators on X, $\mathbb{K}_B(X)$ or $\mathbb{K}(X)$ for the C*-algebra of compact operators on X. The precise definitions are in Section 2.2.

Several types of tensor products are used throughout the notes: $\otimes$ stands for the tensor product of Hilbert modules, $\otimes^{\mathrm{alg}}$ stands for the algebraic tensor product, and $\otimes^{\mathrm{h}}$ stands for the Haagerup tensor product of operator spaces. We will omit the balancing of elements in a balanced tensor product. That is, given an internal tensor product Hilbert module $X \otimes_B Y$, $x \in X$ and $y \in Y$, we will write $x \otimes y$ for the corresponding element in $X \otimes_B Y$. Similar notations are used for operators acting on balanced tensor products as well.

An unbounded operator D on a Hilbert module X with domain $\mathrm{Dom}\, D$ will be denoted by $D\colon X \supseteq \mathrm{Dom}\, D \to X$.

Through this note, all C*-algebras are assumed to be *ungraded*, though all the results can be extended to $\mathbb{Z}/2$-graded C*-algebras, which are the original concern of Kasparov.

1 Index theory in geometry

1.1 Index theorems

One of the most prominent mathematical achievements of the 20th century is the Atiyah–Singer index theorem. It states that the index of an elliptic operator on a compact manifold, which is *analytic* in nature, can be computed via a *topological* formula involving only the characteristic classes of the underlying manifold. The Atiyah–Singer index theorem recovers the Gauss–Bonnet theorem, the Riemann–Roch theorem, and various other classical theorems, as special cases.

An index theorem is an equality between, on the one side, the (Fredholm) index of a (possibly unbounded) Fredholm operator, and on the other side a topological formula computing this index. The equality showcases that the index is topological in nature. Viewed more abstractly, this leads to the notion of an *index pairing*, interpreting the index of an operator as the pairing of a K-homology class and a K-theory class.

1.1.1 Bounded Fredholm operators

In this section we present some definitions and fundamental results on Fredholm operators in $\mathbb{B}(H)$.

Definition 1.1. Let H and K be Hilbert spaces. A closed operator $D\colon H \supseteq \operatorname{Dom} D \to K$ with densely defined adjoint D^* is *Fredholm*, if both $\ker D$ and $\ker D^*$ are finite dimensional. The *index* of a Fredholm operator is defined as

$$\operatorname{Ind} D := \dim\ker D - \dim\ker D^*.$$

A Fredholm operator automatically has closed range (reference).

Example 1.2. If H and K are finite dimensional vector spaces every linear map $D\colon H \to K$ is Fredholm. By the dimension theorem Ind $D = \dim H - \dim K$.

Example 1.3. Let $H = K = \ell^2(\mathbb{N})$ with orthonormal basis e_i defined by $e_i(n) := \delta_{in}$. The unilateral shift $S\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by $S(e_i) := e_{i+1}$. Then S is Fredholm and $\operatorname{Ind} S = -1$.

For a Hilbert space H we denote by $\mathbb{B}(H)$ the C^*-algebra of bounded operators on H and by $\mathbb{K}(H)$ the ideal of compact operators. It is well-known that $\mathbb{K}(H)$ is the closure of the $*$-algebra $\operatorname{Fin}(H)$ of finite rank operators. Both $\mathbb{K}(H)$ and $\operatorname{Fin}(H)$ are ideals in $\mathbb{B}(H)$. The *Calkin algebra* is the quotient $\mathcal{Q}(H) := \mathbb{B}(H)/\mathbb{K}(H)$. Thus by definition of $\mathcal{Q}(H)$ there is an exact sequence of C^*-algebras

$$0 \to \mathbb{K}(H) \to \mathbb{B}(H) \xrightarrow{q} \mathcal{Q}(H) \to 0.$$

We denote by $\operatorname{Fred}(H)$ the set of all Fredholm operators in $\mathbb{B}(H)$.

Theorem 1.4 (Atkinson)**.** *An operator $F \in \mathbb{B}(H)$ is Fredholm if and only $q(F) \in \mathcal{Q}(H)$ is invertible. In particular* $\operatorname{Fred}(H) \subset \mathbb{B}(H)$ *is an open subset.*

We view $\operatorname{Fred}(H)$ as a toplogical space in the relative topology inherited from $\mathbb{B}(H)$. The following theorem is the first fundamental link between Fredholm operators and topology. By a *homotopy* of Fredholm operators we mean a continuous map $F : [0,1] \to \operatorname{Fred}(H)$. The operators $F(0)$ and $F(1)$ associated with the endpoints of such a path are then said to be homotopic. Since paths can be reversed and concatenated, homotopy of Fredholm operators is an equivalence relation.

Theorem 1.5. *The index of a Fredholm operator is invariant under homotopies of Fredholm operators. In particular* $\operatorname{Ind} : \operatorname{Fred}(H) \to \mathbb{Z}$ *is a locally constant function.*

1.1.2 The Gohberg–Krein index theorem

We proceed by discussing a simple case of an index theorem. The *Hardy space* on the circle is

$$H^2(S^1) := \left\{ f \in L^2(S^1) \;\middle|\; f = \sum_{n\geq 0} a_n \mathrm{e}^{2\pi i n x} \right\} \simeq \ell^2(\mathbb{N}).$$

The *Hardy projection* is given by

$$P\colon L^2(S^1) \to H^2(S^1), \qquad P\left(\sum_{n\in\mathbb{N}} a_n \mathrm{e}^{2\pi i n x}\right) = \sum_{n\geq 0} a_n \mathrm{e}^{2\pi i n x}.$$

The *Toeplitz operator* with symbol $f \in \mathrm{C}(S^1)$ is the operator

$$T_f \colon H^2(S^1) \to H^2(S^1), \qquad \phi \mapsto PfP\phi.$$

If $f \in \mathrm{C}(S^1, \mathbb{C}^\times)$ is a nonvanishing function, the associated multiplication operator in $\mathbb{B}(L^2(S^1))$ is invertible. This property is not preserved for the Toeplitz operators T_f. We have the following:

Theorem 1.6 (Gohberg–Krein)**.** *For every $f \in \mathrm{C}(S^1, \mathbb{C}^\times)$, the operator $T_f \colon H^2(S^1) \to H^2(S^1)$ is Fredholm, and*

$$\operatorname{Ind} T_f = -\operatorname{wind}(f),$$

where wind(f) *is the* winding number *of f. If f is moreover C^1, then* $\operatorname{wind}(f) = \int_{S^1} \frac{f'(z)}{f(z)} \mathrm{d}z$.

In this theorem all the ingredients of general index theory come together. The index of the Toeplitz operators T_f is a topologival invariant, and if the function f is differentiable, the index can be computed by the integral of a differential form on S^1.

1.1.3 The Atiyah–Singer index theorem

Now we shall work with elliptic operators on compact manifolds and formulate the famous Atiyah–Singer index theorem. Let M be a compact n-dimensional manifold. Let $S_\pm \to M$ be smooth vector bundles. An operator

$$D_+ \colon \Gamma^\infty(M, S_+) \to \Gamma^\infty(M, S_-)$$

is called *first order*, if locally it is of the form

$$D_+ = a_0(x) + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}.$$

We say D is of *Dirac type*, if:

- For every $f \in \mathrm{C}^\infty(M)$, we have $[D, f] = \mathrm{c}(\mathrm{d}f)$ is the *Clifford multiplication* by $\mathrm{d}f$.

Note that Dirac type operators are automatically elliptic.

Example 1.7. Let M be a Riemannian manifold. Set $S_+ := \Lambda^{\text{even}} T^*M$, $S_- := \Lambda^{\text{odd}} T^*M$, and

$$D_+ := \mathrm{d} + \mathrm{d}^* \colon S_+ \to S_-.$$

A Dirac type operator can be twisted by a connection on another vector bundle as follows. Let $E \to M$ be another vector bundle. Let

$$\nabla \colon \Gamma^\infty(M, E) \to \Gamma^\infty(M, E) \otimes \Omega^1(M)$$

be a connection, that is, it satisfies the following equality:

$$\nabla(\psi f) = \nabla(\psi) f + \psi \otimes \mathrm{d}f, \quad \text{for all } f \in \mathrm{C}^\infty(M) \text{ and } \psi \in \Gamma^\infty(M, E).$$

Define the *twisted Dirac type operator* $1 \otimes_\nabla D_+ \colon \Gamma^\infty(M, E \otimes S_+) \to \Gamma^\infty(M, E \otimes S_-)$:

$$(1 \otimes_\nabla D_+)(e \otimes \psi) := e \otimes D_+\psi + \nabla(e)\psi.$$

Thus, given a Dirac type operator, this construction associates to a vector bundle with connection a new Dirac type operator.

Proposition 1.8. *The operator $1 \otimes_\nabla D_+ \colon \Gamma^\infty(M, E \otimes S_+) \to \Gamma^\infty(M, E \otimes S_-)$ is of Dirac type and extends to a Fredholm operator $L^2(M, E \otimes S_+) \to L^2(M, E \otimes S_-)$.*

In particular D_+ itself is Fredholm. The index thus gives a numerical invariant of such twisted Dirac operators. The index theorem gives a topological formula to compute this invariant.

Theorem 1.9 (Atiyah–Singer)**.** *Let M be a compact manifold. Let $D_+\colon \mathrm{C}^\infty(M,S_+) \to \mathrm{C}^\infty(M,S_-)$ be a Dirac type operator and $E \to M$ be a vector bundle. Let $\nabla\colon \Gamma^\infty(M,E) \to \Gamma^\infty(M,E)\otimes\Omega^1(M)$ be any connection. Then*

$$\mathrm{Ind}(1\otimes_\nabla D_+) = \int_M \mathrm{Ch}(E)\hat{\mathrm{A}}(S),$$

where $\mathrm{Ch}(E)$ *and* $\hat{\mathrm{A}}(S)$ *are characteristic classes of E and $S = S_+ \oplus S_-$.*

Already here it is important to note that the cohomology class $\mathrm{Ch}(E)$ depends only on the vector bundle E and not on the chosen connection ∇. Although the connection is essential in the construction of the operator $1\otimes_\nabla D_+$, any other connection will yield the same index.

Example 1.10. Let $D = \mathrm{d}+\mathrm{d}^*$ act on $L^2(M,\Lambda^*T^*M)$. Then the Atiyah–Singer index theorem for D is

$$\mathrm{Ind}(\mathrm{d}+\mathrm{d}^*) = \chi(M) = (2\pi)^{-n/2}\int_M \mathrm{Pf}(-R),$$

where $\chi(M)$ is the Euler characteristic of M, $\mathrm{Pf}(-R)$ is the Pfaffian of the Riemannian curvature tensor R.

In particular, if M is a surface. Then

$$\chi(M) = \frac{1}{2\pi}\int_M r_M(x)\mathrm{d}x,$$

where $r_M(x)$ is the scalar curvature at x. This is the classical Gauss–Bonnet theorem.

1.1.4 Atiyah's observation

The Atiyah–Singer index theorem provides a geometric method for computing the index of $1\otimes_\nabla D$, which is a topological invariant. Vector bundles over a compact manifold generate the K-theory group $\mathrm{K}^0(M)$. Atiyah realised that the index $\mathrm{Ind}(1\otimes_\nabla D)$ should be viewed as a pairing

$$([E],[D]) \mapsto \langle E, D\rangle := \mathrm{Ind}(1\otimes_\nabla D_+) \in \mathbb{Z}.$$

So he suggested that elliptic operators over M should generate a "dual" group of $\mathrm{K}^0(M)$: the K-*homology* of M. The index pairing is just a map

$$\mathrm{K}^0(M)\times \mathrm{K}_0(M) \to \mathbb{Z}, \qquad (E,D)\mapsto \mathrm{Ind}(1\otimes_\nabla D).$$

Here $[E]$ and $[D]$ denote equivalence classes of vector bundles and operators under a certain relation. For vector bundles this relation is dictated by topological K-theory. Identifying the correct relation for operators $[D]$ was at the heart of the search for the right definition of the dual theory, K-homology, and eventually, KK-theory.

From a functoriality perspective, the index pairing should be compatible with the Chern characters. The Chern character is a rational isomorphism:

$$\mathrm{Ch}\colon\ \mathrm{K}^0(M) \to \mathrm{H}^{\mathrm{even}}(M;\mathbb{Q}) := \bigoplus_i \mathrm{H}^{2i}(M;\mathbb{Q}),$$

The index pairing between K-theory and K-homology makes the following diagram commute:

$$\begin{array}{ccccc}
\mathrm{K}^0(M) & \times & \mathrm{K}_0(M) & \longrightarrow & \mathbb{Z} \\
\downarrow{\scriptstyle \mathrm{Ch}} & & \downarrow{\scriptstyle \mathrm{Ch}} & & \downarrow \\
\mathrm{H}^{\mathrm{even}}(M;\mathbb{Q}) & \times & \mathrm{H}_{\mathrm{even}}(M;\mathbb{Q}) & \longrightarrow & \mathbb{Q}.
\end{array}$$

The index is a topological invariant and index theorems allow for the computation of this invariant through geometric methods.

1.2 C^*-algebras and KK-theory

We broaden our scope from topology to C^*-algebras. The operator theoretic formulation of K-theory is, in essence, a C^*-algebraic construction. It allows us to apply index theory in different contexts, such as dynamics and representation theory.

1.2.1 Noncommutative geometry and C^*-algebras

A C^*-algebra is a closed *-subalgebra of $\mathbb{B}(H)$, the bounded operators on some Hilbert space H. If X is a locally compact Hausdorff space, then $\mathrm{C}_0(X)$ is a C^*-algebra. The Gelfand–Naimark theorem states that all commutative C^*-algebras are of this form:

Theorem 1.11 (Gelfand–Naimark)**.** *Any commutative* C^**-algebra is* **-isomorphic to* $\mathrm{C}_0(X)$ *for some locally compact Hausdorff space* X*.*

Now let X be a compact Hausdorff space and $E \to X$ be a (finite-dimensional, locally trivial) complex vector bundle. The space of continuous sections $\Gamma(M, E)$ is a finitely generated, projective $\mathrm{C}(X)$-module. The Serre–Swan theorem says that all finitely generated, projective $\mathrm{C}(X)$-modules arise in this way:

Theorem 1.12 (Serre–Swan)**.** *Let* X *be a compact Hausdorff space. Every finitely generated, projective* $\mathrm{C}(X)$*-module is isomorphic to* $\Gamma(M, E)$ *for some finite-dimensional, locally trivial complex vector bundle* $E \to X$*.*

Suggested by these theorems, noncommutative C^*-algebras should be viewed as as a generalisation of topological spaces, and projective modules as a generalisation of vector bundles. This is the starting point of noncommutative geometry. As motivation, certain *commutative* geometry gives rise to some natural noncommutative C^*-algebras. An important equivalence relation among C^*-algebras is *Morita equivalence.*

Recall that for Γ a discrete group, the reduced group C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ is the closure of the group algebra $\mathbb{C}[\Gamma]$ in its left regular representation on $\mathbb{B}(\ell^2(\Gamma))$, where elements of $\mathbb{C}[\Gamma]$ act by convolution. Group C^*-algebras are special cases of *crossed products*: if Γ acts on a locally compact Hausdorff space X by homeomorphisms, then we can define a C^*-algebra

$$\mathrm{C}_0(X) \rtimes \Gamma$$

as the closure of the twisted group algebra $\mathbb{C}[\Gamma, \mathrm{C}_0(X)]$ in $\mathbb{B}(L^2(X \times \Gamma))$. Group C^*-algebras correspond to the special case where X is a point. If Γ is non-abelian or the action of Γ on X is non-trivial, then the C^*-algebra $\mathrm{C}_0(X) \rtimes \Gamma$ is noncommutative. Nevertheless, we have

Theorem 1.13. *Suppose that* Γ *acts freely and properly on a locally compact Hausdorff space* X*. Then the* C^**-algebras* $\mathrm{C}_0(X/\Gamma)$ *and* $\mathrm{C}_0(X) \rtimes \Gamma$ *are Morita equivalent.*

Example 1.14. If $X = \Gamma$ and Γ acts on X by translation. Then $\mathrm{C}_0(X/\Gamma) = \mathbb{C}$ and $\mathrm{C}_0(X) \rtimes \Gamma \simeq \mathbb{K}(L^2(G))$.

Morita equivalent C^*-algebras share many similar properties. For instance, there is an intimate relation between their categories of Hilbert space representations and they have the same K-theory. The previous example shows that $\mathrm{C}_0(X) \rtimes \Gamma$ could be viewed as a "noncommutative quotient" of X under G. In fact, many properties of this action (i.e. the C^*-dynamical system $(\mathrm{C}_0(X), G, \alpha)$) are reflected by corresponding properties of $\mathrm{C}_0(X) \rtimes G$. This should clarify why noncommutative C^*-algebras are also useful in the study of "commutative" geometry.

1.2.2 K-theory and K-homology

The K-theory of a compact Hausdorff space X is generated by vector bundles $E \to X$. From the viewpoint of the Swan–Serre theorem, vector bundles over X are finitely generated, projective $\mathrm{C}(X)$-modules. So we immediately obtain a definition of K-theory of a unital C^*-algebra A: it is the abelian group generated by equivalence classes of finitely generated projective A-modules. We write $\mathrm{K}_0(A)$ for the K-theory of A.

As suggested by Atiyah, a theory *dual* (in the sense of cohomology) to K-theory should be modelled on elliptic operators. As alluded to before, obtaining the correct notion of cycle and the formulation of the precise equivalence relation was a nontrivial task, ultimately completed by Kasparov [Kas75].These cycles are now called Fredholm modules or Kasparov K-cycles, and are modelled on elliptic operators. However, the latter are usually unbounded, and Fredholm modules are the analogues of the *bounded transforms* of such elliptic operators (c.f. Section **??**).

Shortly thereafter, the advent of Connes' noncommutative geometry [Con94] gave rise to an approach modelling K-cycles on unbounded elliptic operators directly.

Definition 1.15. A spectral triple $(\mathcal{A}, H, D)$ over a C^*-algebra A consists of:

- A $\mathbb{Z}/2$-graded Hilbert space $H = H_+ \oplus H_-$ such that A is represented by even bounded operators on H.
- $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ for a closed operator $D_+ : H_+ \to H_-$ and $D_- := D_+^*$, such that $a(1+D^2)^{-1}$ is compact for all $a \in A$.
- A dense subalgebra $\mathcal{A} \subseteq A$ such that for all $a \in \mathcal{A}$: a maps $\operatorname{Dom} D_+$ to $\operatorname{Dom} D_+$ and $[D_+, a]$ extends to a bounded operator on H.

Example 1.16. Let M be a compact manifold. Then $(\mathrm{C}^\infty(M), L^2(M, \Lambda^* T^* M), \mathrm{d}+\mathrm{d}^*)$ is a spectral triple on $\mathrm{C}(M)$.

Definition 1.17. The K-homology of A, denoted by $\mathrm{K}^0(A)$, is the abelian group generated by homotopy equivalence classes of spectral triples over A.

We refer to [DM20] for a precise definition of homotopy of spectral triples, and more generally, of unbounded Kasparov modules.

In the view of K-homology and K-theory, the index pairing can be defined algebraically as follows. Given a spectral triple $(\mathcal{A}, H, D)$ the $\mathcal{A}$-bimodule

$$\Omega^1_D := \left\{ \sum a_i [D, b_i] \mid a_i, b_i \in \mathcal{A} \right\}$$

is the *noncommutative differential 1-forms.* A precise definition is given in Section 4.2.1. In the case of a manifold M and $(\mathcal{A}, H, D) := (\mathrm{C}^\infty(M), L^2(M, S), D)$, Ω^1_D recovers the the module of 1-forms $\Omega^1(M)$ in its Clifford representation.

Subsequently, under the assumption that $\mathcal{A}$ is *stable under holomorphic functional calculus in* $\mathcal{A}$ any finitely generated projective module P over A is, up to isomorphism, the completion of a finitely generated projective module $\mathcal{P}$ over $\mathcal{A}$. Such projective modules admit connections

$$\nabla : \mathcal{P} \to \mathcal{P} \otimes_{\mathcal{A}} \Omega^1_D, \quad \nabla(pa) = \nabla(p)a + p \otimes [D, a].$$

Given such a connection ∇ one forms the densely defined unbounded operator

$$1 \otimes_\nabla D \colon \mathcal{P} \otimes^{\mathrm{alg}}_{\mathcal{A}} \operatorname{Dom} D \to P \otimes_A H, \quad 1 \otimes_\nabla D(p \otimes h) := p \otimes Dh + \nabla(p)h.$$

As in the geometric setup, this operator turns out to be essentially self-adjoint and Fredholm. The index pairing between K-theory and K-homology is then given by

$$\mathrm{K}^0(A) \times \mathrm{K}_0(A) \to \mathbb{Z}, \qquad (D, P) \mapsto \operatorname{Ind}(1 \otimes_\nabla D).$$

1.2.3 Properties of KK-theory

After providing the correct equivalence relations on Fredholm modules, and consequently a correct analytic model for K-homology in [Kas75], Kasparov realised that K-theory and K-homology can be combined into a bivariant cohomology theory for C^*-algebras. This is Kasparov's bivariant K-theory, or KK-theory, c.f. [Kas80].

For every pair of separable C^*-algebras A and B, Kasparov assigned to it a $\mathbb{Z}/2$-graded abelian group $\mathrm{KK}_*(A, B) = \mathrm{KK}_0(A, B) \oplus \mathrm{KK}_1(A, B)$, as the equivalence classes of Kasparov modules, such that:

- $\mathrm{KK}_*(\mathbb{C}, A) \simeq \mathrm{K}_*(A)$ is the K-theory of A.
- $\mathrm{KK}_*(A, \mathbb{C}) \simeq \mathrm{K}^*(A)$ is the K-homology of A.
- There exists an associative, bilinear product

$$\mathrm{KK}_i(A, B) \times \mathrm{KK}_j(B, C) \to \mathrm{KK}_{i+j}(A, C),$$

  called the *Kasparov product.*
- The Kasparov product

$$\mathrm{KK}_0(\mathbb{C}, A) \times \mathrm{KK}_0(A, \mathbb{C}) \to \mathrm{KK}_0(\mathbb{C}, \mathbb{C}) \simeq \mathrm{K}_0(\mathbb{C}) \simeq \mathbb{Z}$$

  recovers the *index pairing.*
- Every $x \in \mathrm{KK}_i(A, B)$ gives a map $\mathrm{K}_*(A) \to \mathrm{K}_{*+i}(B)$.
- $\mathrm{KK}_0(A, A)$ is a ring and $\mathrm{K}^*(A)$ and $\mathrm{K}_*(A)$ are modules over this ring.

Since the index pairing is a special case of the Kasparov product, it turns out that KK-theory is a natural framework for studying index theory. However, the construction of the Kasparov product is highly non-trivial: though the product exists uniquely up to equivalence (thanks to the delicate construction of Kasparov), it is often hard to construct directly. In practice one often empoys a "guess-and-check" procedure, which we outline in Section 4.

Kasparov's original definitions use the notion of bounded Fredholm modules, and bounded adjointable operators on Hilbert C^*-modules. This approach has the advantage that functional analytic difficulties can be dealt with rigorously in the context of bounded operators. This comes at the expense of losing touch with the algebraic and geometric formulae that aid computation in the classical case. The problem here is that these formulae almost always involve operators that are unbounded in a functional analytic context.

Just as K-homology allows for unbounded K-cycles (spectral triples), it is possible to represent KK-theory classes by *unbounded Kasparov modules.* The advantages of unbounded Kasparov modules is that one can construct the Kasparov product explicitly, using a geometrically inspired notion of *connection*, c.f. Section 4.2.1 and [KL13, Mes14, MR16]. These operators are closer to the operators that naturally arise in geometry and index theory, and the unbounded picture allows one to circumvent the use of their bounded transforms. The disadvantage is that one has to now deal with the subtleties of unbounded operators and their domains, to safeguard that algebraic formulae make good sense. This leads to the theory of *unbounded regular* operators on Hilbert C^*-modules. The formulation of the equivalence relation at the unbounded level has only recently been addressed more thoroughly [DGM18, Kaa20, DM20].

Unbounded KK-theory has seen a revival and rapid development in the last two decades. It provides a natural framework for *noncommutative differential geometry* à la Connes [Con85].

2 KK-theory

In this section we will define the cycles for KK-theory and the equivalence relation on them. The equivalence classes turn out to form an abelian group, the KK-group, and we discuss its relation to topological K-theory.

2.1 K-theory and Fredholm operators

In this section we provide some informal intuition for the main definitions in KK-theory. The cycles of KK-theory are Kasparov modules, modelled on Fredholm operators acting on Hilbert C^*-modules. This definition is motivated by the close relation between Fredholm operators and toplogical K-theory, as shown by the Atiyah–Jänich theorem:

Theorem 2.1 (Atiyah–Jänich)**.** *Let X be a compact Hausdorff space and H a seperable infinite dimensional Hilbert space. Then $\mathrm{K}^0(X) \simeq [X, \mathrm{Fred}(H)]$, the set of homotopy classes of maps $X \to \mathrm{Fred}(H)$.*

Sketch of proof. The idea of the proof (c.f. [Ati67, Appendix]) is as follows. A map $X \to \mathrm{Fred}(H)$ gives a "continuous field" of Fredholm operators $\{F_x\}_{x\in X}$ indexed by X. Every F_x is a Fredholm operator and has finite dimensional kernel and cokernel. We may attempt to build vector bundles $\{\ker F_x\}_{x\in X}$ and $\{\mathrm{coker}\, F_x\}_{x\in X}$ and build a class $[\ker F_x] - [\mathrm{coker}\, F_x] \in \mathrm{K}^0(X)$. This is not quite true, because the dimensions of $\ker F_x$ and $\mathrm{coker}\, F_x$ might jump with respect to $x \in X$. Nevertheless, the dimension jump will happen simultaneously to $\ker F_x$ and $\mathrm{coker}\, F_x$ because the Fredholm index is homotopy invariant. Therefore, with a modification of the bundles $\ker F_x$ and $\mathrm{coker}\, F_x$ by taking these dimension jumps into account, one obtains a well-defined class in $\mathrm{K}^0(X)$. □

Consider the trivial bundle of Hilbert spaces $X \times H \to X$ over X. The space of sections of this bundle can be identified with the space

$$E := C(X, H) := \{\phi : X \to H : \phi \text{ continuous } \}$$

of continuous maps $X \to H$. The space E is naturally a $C(X)$-module by setting $\phi \cdot f(x) := \phi(x)f(x)$, for $\phi \in E$ and $f \in C(X)$. The inner product $\langle\cdot,\cdot\rangle_H$ on H defines a pairing $E \times E \to C(X)$ via

$$\langle \phi, \psi\rangle(x) := \langle \phi(x), \psi(x)\rangle_H. \tag{1}$$

These structures make E into a *Hilbert C^*-module* over $C(X)$, a notion that will be defined formally in the next section.

A continuous map $F : X \to \mathrm{Fred}(H) \subset B(H)$ can now be viewed as an operator $F : E \to E$ via $(F\phi)(x) := F(x)\phi(x)$. Up to homotopy, we can choose F such that

$$1_E - F^*F, \quad 1_E - FF^* \in C(X, \mathbb{K}(H)),$$

that is, they are pointwise compact pertubations of the identity operator on E. The pair

$$\left(E \oplus E, \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}\right),$$

is then an example of an *even Kasparov module* (see Definition 2.20 below).

2.2 Hilbert C^*-modules

We proceed with the formal definitions that will allow us to make the observations form the previous subsection rigorous. Standard references for Hilbert C^*-modules are the books [Lan95, MT05].

Definition 2.2. Let B be a C^*-algebra. A Hilbert C^*-module over B (or a Hilbert B-module) is a right B-module X equipped with a sesquilinear pairing $\langle\cdot,\cdot\rangle : X \times X \to B$, such that for all $x, y \in X$ and $b \in B$:

1. $\langle x, yb\rangle = \langle x, y\rangle b$.

2. $\langle x, y\rangle^* = \langle y, x\rangle$.
3. $\langle x, x\rangle \geq 0$ in B, and $\langle x, x\rangle = 0$ iff $x = 0$.
4. X is complete in the norm $\|x\|^2 = \|\langle x, x\rangle\|_B$.

Example 2.3. A Hilbert space H is a Hilbert $\mathbb{C}$-module.

Example 2.4. Let X be a (locally) compact Hausdorff space and $\mathcal{H} \xrightarrow{\pi} X$ a continuous field of Hilbert spaces over X. Then the space of continuous sections $\Gamma(X, \mathcal{H})$ is a Hilbert C^*-module over $C_0(X)$ in the inner product $\langle s, t\rangle(x) := \langle s(x), t(x)\rangle_{\mathcal{H}_x}$. Here $\mathcal{H}_x$ denotes the fiber $\pi^{-1}(x)$ which is a Hilbert space. In fact every Hilbert C^*-module over $C_0(X)$ arises in this way. The case of a locally trivial finite dimensional vector bundle $\mathcal{E} \xrightarrow{\pi} X$ equipped with a Riemannian metric is a special case of this construction.

Example 2.5. A C*-algebra A is a Hilbert A-module with $\{a, b\} := a^*b$. More generally, for $n \in \mathbb{N}$ the rank n free module $A^n := \bigoplus_{k=1}^n A$ is a Hilbert C^*-module in the inner product

$$\langle (a_k), (b_k)\rangle := \sum_{k=1}^n a_k^* b_k.$$

Example 2.6. Let (X, μ) be a measure space and A a C^*-algebra. Denote by

$$L^2(X, A) := \left\{ f : X \to A : \int_X f(x)^* f(x) \mathrm{d}\mu < \infty \right\}.$$

Note that the condition of f is that the relevant integral converges in A. This condition on f is weaker than requiring that

$$\int_X \|f(x)\|_A^2 \mathrm{d}\mu < \infty. \tag{2}$$

If f is such that (2) holds then since $f(x)^* f(x) \leq \|f(x)\|^2$ in (the unitsation of) A, we have

$$\int_X f(x)^* f(x) \mathrm{d}\mu < \int_X \|f(x)\|_A^2 \mathrm{d}\mu < \infty,$$

but the reverse implication does not hold.

Definition 2.7. Let X be a Hilbert B-module. The set of *adjointable operators* on X is

$$\mathrm{End}_B^*(X) := \{T \colon X \to X \mid \exists T^* \colon X \to X, \text{ s.t. } \langle Tx, y\rangle = \langle x, T^* y\rangle \text{ for all } x, y \in X\}.$$

It is noteworthy that the above algebraic definition implies several natural properties.

Proposition 2.8. *If $T \in \mathrm{End}_B^*(X)$ then T is bounded and B-linear and $\mathrm{End}_B^*(X)$ is a C^*-algebra in the operator norm.*

Proof. To see that T is bounded consider $\{T_x : X \to B\}_{\|x\| \leq 1}$, the family of maps given by

$$T_x : X \to B, \quad T_x(y) := \langle x, y\rangle.$$

Then for all $y \in X$ we have

$$\sup_{\|x\| \leq 1} \|T_x(y)\| = \sup_{\|x\| \leq 1} \|\langle Tx, y\rangle\| = \sup_{\|x\| \leq 1} \|\langle x, T^* y\rangle\| \leq \|T^* y\| < \infty.$$

By the uniform boundedness principle $\sup_{\|x\| \leq 1} \|T_x\| < \infty$. Now $\|T_x\| = \|Tx\|$ as can be seen via

$$\|T_x\| = \sup_{\|y\| \leq 1} \|\langle Tx, y\rangle\| \leq \|Tx\|,$$

and

$$\|T_x\| = \sup_{\|y\|\leq 1} \|\langle Tx, y\rangle\| \geq \left\| \left\langle Tx, \frac{Tx}{\|Tx\|} \right\rangle \right\| = \|Tx\|.$$

For B-linearity we consider

$$\begin{aligned}\langle (Tx)b - T(xb), y\rangle = \langle (Tx)b, y\rangle - \langle T(xb), y\rangle &= b^*\langle Tx, y\rangle - \langle T(xb), y\rangle \\ &= b^*\langle x, T^*y\rangle - \langle xb, T^*y\rangle \\ &= b^*\langle x, T^*y\rangle - b^*\langle x, T^*y\rangle = 0,\end{aligned}$$

from which we deduce that $(Tx)b - T(xb) = 0$, so T is B-linear. To see that $\mathrm{End}^*_B(X)$ is a C^*-algebra, suppose that $T_n \in \mathrm{End}^*_B(X)$ converges in operator norm to some bounded operator T. Since $\|A^*\| = \|A\|$ for all $A \in \mathrm{End}^*_B(X)$ the sequence T^*_n is Cauchy and thus converges to a bounded operator S. Then

$$\langle Tx, y\rangle = \lim\langle T_n x, y\rangle = \lim\langle x, T^*_n y\rangle = \langle x, Sy\rangle,$$

so T is adjointable and $S = T^*$. The C^*-identity follows in the same way as in Hilbert space. □

Definition 2.9. Let X be a Hilbert B-module and $x, y \in X$. Define $T_{x,y}$ as the rank-one operator on X:

$$T_{x,y}(z) := x\langle y, z\rangle.$$

It is straightforward to check that $T^*_{x,y} = T_{y,x}$ so that $T_{x,y}$ is adjointable. Moreover $T_{x,y}T_{z,w} = T_{x,\langle y,z\rangle w}$, so the set of rank one operators is closed under multiplication and the $*$-operation. It is not closed under addition. We define the set of *finite rank operators* on X by

$$\mathrm{Fin}_B(X) := \mathrm{span}_{\mathbb{C}} \{T_{x,y} : x, y \in X\},$$

and this is a $*$-algebra and a two-sided ideal in $\mathrm{End}^*_B(X)$. However, $\mathrm{Fin}_B(X)$ is not closed in the operator norm topology. Its norm closure $\mathbb{K}_B(X)$ is the C^*-algebra of *compact operators* on X.

Lemma 2.10. *The C^*-algebra $\mathbb{K}_B(X)$ is a closed, two-sided ideal of* $\mathrm{End}^*_B(X)$.

In the case that $X = B$ there is are $*$-isomorphisms $\mathbb{K}_B(X) \simeq B$ and $\mathrm{End}^*_B(X) \simeq M(B)$, where the latter denotes the *multiplier algebra* of B. In fact this approach can be used as the definition of $M(B)$.

2.3 Bimodules and tensor products

We now arrive at the appropriate notion of bimodule in the C^*-context.

Definition 2.11. Let A and B be C*-algebras. By a C^*-correspondence for (A, B), we mean a right Hilbert B-module X together with a $*$-homomorphism $\pi\colon A \to \mathrm{End}^*_B(X)$.

We sometimes oppress the $*$-homomorphism $\pi\colon A \to \mathrm{End}^*_B(X)$ in the notation, although it is often useful to keep track of it.

Example 2.12. A $*$-homomorphism $\pi : A \to M(B)$ gives a C^*-correspondence for (A, B) by setting $X = B$ with its standard C^*-module structure.

Let X be a Hilbert B-module and Y be a (B, C)-bimodule. The algebraic tensor product $X \otimes^{\mathrm{alg}}_B Y$ is equipped with a C-valued inner product:

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle := \langle y_1, \langle x_1, x_2\rangle y_2\rangle. \tag{3}$$

This satisfies conditions 1–3 in the definition of a Hilbert C*-module. Although it is straightforward to check that the inner product (3) is well-defined on $X \otimes^{\mathrm{alg}}_B Y$, it is also *nondegenerate* (see [Lan95, Proposition 4.5]). To get a Hilbert C^*-module over C it thus suffices to complete $X \otimes^{\mathrm{alg}}_B Y$ under the norm induced by this inner product:

Definition 2.13. The *interior tensor product* $X \otimes_B Y$ is the completion of $X \otimes_B^{\mathrm{alg}} Y$ under the inner product (3). We write $X \otimes_\pi Y$ if we want to emphasize the representation $\pi\colon B \to \mathrm{End}_C^*(Y)$.

Given X and Y as above, an operator $T \in \mathrm{End}_B^*(X)$ defines a right C-module map

$$T \otimes 1 : X \otimes_B^{\mathrm{alg}} Y \to X \otimes_B^{\mathrm{alg}} Y, \quad x \otimes y \mapsto Tx \otimes y. \tag{4}$$

Proposition 2.14. *The map $T \mapsto T \otimes 1$ via (4) extends to a $*$-homomorphism $\Pi : \mathrm{End}_B^*(X) \to \mathrm{End}_C^*(X \otimes_\pi Y)$. When π is injective, then so is Π. If $\pi : B \to \mathbb{K}_C(Y)$ then $\Pi : \mathbb{K}_B(X) \to \mathbb{K}_C(X \otimes_\pi Y)$.*

Given an (A, B)-correspondence X and a (C, D)-correspondence Y, there is a second notion of *exterior* tensor product that will play a less important role in these notes, but we introduce it here for completion. First form the algebraic tensor product $X \otimes_{\mathbb{C}}^{\mathrm{alg}} Y$ of vector spaces and view it as a right $B \otimes_{\mathbb{C}}^{\mathrm{alg}} D$-module and a left $A \otimes_{\mathbb{C}}^{\mathrm{alg}} C$-module. The formula

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle := \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle \in B \otimes_{\mathbb{C}}^{\mathrm{alg}} D,$$

defines an inner product. Completion of $X \otimes_{\mathbb{C}}^{\mathrm{alg}} Y$ via the minimal C^*-completion of $B \otimes_{\mathbb{C}}^{\mathrm{alg}} D$ will give a Hilbert C^*-module $X \overline{\otimes} Y$. The obvious left action of $A \otimes_{\mathbb{C}} C$ on $X \otimes_{\mathbb{C}}^{\mathrm{alg}} Y$ extends to a $*$-homomorphism $A \overline{\otimes} C \to \mathrm{End}_{B \overline{\otimes} D}^*(X \overline{\otimes} Y)$.

Definition 2.15 (Standard Hilbert modules)**.** Let H be a separable Hilbert space and B a C^*-algebra. The *standard Hilbert B-module* over H is the completed tensor product $H \otimes B$.

In this definition, either the interior or exterior tensor product constructions can be used. The case $H = L^2(X, \mu)$ yields an isomorphism $L^2(X, \mu) \otimes A \simeq L^2(X, A)$ from Example 2.6. Although all seperable Hilbert spaces are isomorphic, so are the modules $H \otimes B$. In practice the choice of particular Hilbert space H may matter for explicit calculations.

2.4 Kasparov's stablisation theorem

There is a close relation between Hilbert C^*-modules and the algebraic notion of projectivity. A finitely generated projective module P over a unital C^*-algebra A admits the structure of a Hilbert C^*-module by embedding P into A^n and restricting the inner product. The converse is true as well.

Theorem 2.16. *A finitely generated Hilbert C^*-module P over a unital C^*-algebra A is algebraically projective.*

This property allows one to view Hilbert C^*-modules as a generalisation of finitely generated projective modules. This viewpoint is made precise by the following remarkable result of Kasparov.

Theorem 2.17. *If X is a countably generated Hilbert B-module. Then there is an isomorphism of Hilbert B-modules*

$$X \oplus H \otimes B \xrightarrow{\sim} H \otimes B,$$

where $H \otimes B$ is the interior tensor product Hilbert B-module $H_{\mathbb{C}} \otimes_{\mathbb{C}} B_B$ for a(ny) separable Hilbert space H.

An abstract existence result, the isomorphism alluded to in the above theorem is non-explicit and non-unique. In practice the following corollary is of use. A an adjointable operator $V : X \to Y$ is an *isometry* if $V^*V = \mathrm{Id}_X$.

Corollary 2.18. *Let X be a countably generated Hilbert B-module and B^+ the unitisation of B. Then there exists an isometry $V\colon X \to H \otimes B^+$.*

Proof. We can view X as a Hilbert C^*-module over B^+ and apply the stabilisation theorem to find V. □

Let $\{e_i\}_{i\in\mathbb{N}}$ be a countable basis of H and set $x_i := V^*(e_i \otimes 1)$ for all $i \in \mathbb{N}$. Then the family $\{x_i\}_{i\in\mathbb{N}}$ is that for all $x \in X$, the equality

$$x = \sum_{i\in\mathbb{N}} x_i \langle x_i, x\rangle$$

holds strongly. The set $\{x_i\}_{i\in\mathbb{N}}$ is a *frame* for X which is tight and normalised in the sense of [FL02]. Frames are not unique. An important technical application of frames is the following.

Recall that on the internal tensor product $X \otimes_B Y$, for $T \in \mathrm{End}^*_B(X)$ the operators $T \otimes 1$ are well-defined. For $S \in \mathrm{End}^*_C(Y)$ the operator $1 \otimes S$ is well-defined on $X \otimes_B Y$ only if S commutes with B.

Proposition 2.19. *Let X be a countably generated Hilbert B-module and $V\colon X \to H \otimes B$ and isometry. For any (B, C) C^*-correspondence Y the map*

$$W : X \otimes_B Y \to H \otimes Y, \quad x \otimes y \mapsto V(x)y,$$

is an isometry. The map

$$\mathrm{End}^*_C(Y) \to \mathrm{End}^*_C(X \otimes Y), T \mapsto W^*(1 \otimes T)W,$$

is a completely positive contraction and

$$W^*(1 \otimes T)W(x \otimes y) = \sum_i x_i \otimes T\langle x_i, x\rangle y,$$

where $x_i := V^*(e_i \otimes 1)$ is any frame associated to a basis of H.

We can thus use frames and isometries to push-forward operators from Y to $X \otimes_B Y$. We emphasise that this push-forward does not give a $*$-homomorphism, but only a linear map.

2.5 KK-theory

We arrive at the Fredholm picture of KK-theory as introduced by Kasparov. We need the notion of a $\mathbb{Z}/2$-grading on a Hilbert module X over B. By definition, a $\mathbb{Z}/2$-grading is an operator $\gamma \in \mathrm{End}^*_B(X)$ with $\gamma = \gamma^*$ and $\gamma^2 = 1$. A $\mathbb{Z}/2$-graded module X decomposes as $X_+ \oplus X_-$, the ± 1-eigenmodules of γ. The grading operator γ also decomposes the endomorphisms $\mathrm{End}^*_B(X)$ into two pieces: an operator T is a called *even* if $\gamma T \gamma = T$ and *odd* if $\gamma T \gamma = -T$.

Definition 2.20. Let A and B be C*-algebras. An *even Kasparov module* is a pair (X, F), where:

- X is a $\mathbb{Z}/2$-graded (A, B)-bimodule;
- $F \in \mathrm{End}^*_B(X)$ is odd operator,

such that:

$$a(F^2 - 1), \quad a(F - F^*), \quad [F, a] \quad \in \quad \mathbb{K}_B(X), \quad \text{for all } a \in A. \tag{5}$$

We write $\mathbb{E}(A, B)$ for the class of all even Kasparov (A, B)-modules.

Definition 2.21. Let (X^0, F^0) and (X^1, F^1) be even Kasparov (A, B)-modules. We say they are *homotopic*, if there exists an even Kasparov $(A, B[0, 1])$-module (X, F), such that

$$(X \otimes_{\mathrm{ev}_0} B, F \otimes 1) \simeq (X^0, F^0) \quad \text{and} \quad (X \otimes_{\mathrm{ev}_1} B, F \otimes 1) \simeq (X^1, F^1).$$

Here we use

$$\mathrm{ev}_t\colon B[0, 1] \to B, \qquad f \mapsto f(t),$$

to equip B with a $(B[0, 1], B)$-bimodule structure.

Remark 2.22. Homotopy is an equivalence relation on $\mathbb{E}(A, B)$.

Definition 2.23. The Kasparov group $\mathrm{KK}(A, B)$ is the set of homotopy equivalence classes of $\mathbb{E}(A, B)$.

Proposition 2.24. $\mathrm{KK}(A, B)$ *is an abelian group. The addition is represented by the direct sum of Kasparov modules* $(X^0, F^0) \oplus (X^1, F^1) = (X^0 \oplus X^1, F^0 \oplus F^1)$. *The zero element is represented by Kasparov modules such that operators in* (5) *are zero.*

Remark 2.25. By removing all the "gradings" in Definition 2.20 — that is, X is an ungraded (A, B)-bimodule and F is ungraded — one obtains the definition of an odd Kasparov module, being a cycle of $\mathrm{KK}_1(A, B)$.

3 Unbounded KK-theory

In this section we introduce the unbounded picture of KK-theory. Although is in some aspects easier and more algebraic, it comes at the expense of dealing with the subtleties of unbounded operators and their domains.

3.1 Unbounded operators on Hilbert C^*-modules

We first recall some basic definitions and facts of unbounded operators on Hilbert C^*-modules. An elaborate reference is [Lan95]. The adjoint of a densely defined operator on a Hilbert C^*-module is defined in the same way a in the Hilbert space setting.

Definition 3.1. Let X_B be a Hilbert B-module. A densely defined operator $D\colon X_B \supseteq \mathrm{Dom}\, D \to X_B$ with densely defined adjoint is *regular*, if the operator $1 + D^*D$ has dense range.

The regularity axiom is automatically satisfied for Hilbert space operators. Examples of non-regular operators in Hilbert C^*-modules exist already over commutative C^*-algebras and can be found in [KL12, Dun22].

We now recall that the *graph* of an operator $D : X \supset \mathrm{Dom}\, D \to Y$ is submodule

$$\mathrm{G}(D) := \left\{ \begin{pmatrix} x \\ Dx \end{pmatrix} : x \in \mathrm{Dom}\, D \right\} \subset X \oplus Y.$$

Definition 3.2. A densely defined operator $D\colon X \to Y$ is *closed* if its graph $\mathrm{G}(D)$ is closed in $X \oplus Y$, or equivalently, if $\mathrm{Dom}\, D$ is closed in the *graph norm* $\|x\|_D := \|x\| + \|Dx\|$. A densely defined operator $\overline{D}$ is called the *closure* of D, if $\mathrm{G}(\overline{D})$ is the closure of $\mathrm{G}(D)$. A *core* for D is a subspace $K \subseteq \mathrm{Dom}\, D$ such that D is the closure of $D|_K$. That is:

$$\mathrm{G}(D) = \overline{\mathrm{G}(D|_K)}.$$

The regularity axiom can be characterised algebraically via the graph $\mathrm{G}(D)$. For a closed submodule $M \subset X$ of a Hilbert C^*-module, its *orthogonal complement* is

$$M^\perp := \{x \in X : \forall m \in M \quad \langle x, m\rangle = 0\}.$$

The map

$$M \oplus M^\perp \to X, \quad (m, n) \mapsto m + n, \tag{6}$$

is an isometry, but may fail to be surjective. To see this consider a compact topological space Z and let $U \subset Z$ be an open dense subset. Then $C_0(U) \subsetneq C(Z)$ is a closed ideal with $C_0(U)^\perp = 0$.

Definition 3.3. A submodule $M \subset X$ is *complemented* if the map (6) is surjective.

Decompositions (6) are in one-to-one correspondence with projections $P = P^2 = P^* \in \operatorname{End}^*_B(X)$. Thus, an immediate corollary of this definition is that a closed submodule $M \subset X$ is complemented if and only if there is a projection $P = P^2 = P^* \in \operatorname{End}^*_B(X)$ such that $M = PX$. Given such a projection we have $M^\perp = (1-P)X$. Below we will see a different, representation theoretic criterion for complementability.

We can now charaterise regular operators in terms of complementability of their graphs. On $X \oplus X$ consider the map

$$v : X \oplus X \to X \oplus X, \quad v(x,y) := (-y, x),$$

which is a unitary isomorphism.

Proposition 3.4. *Let $D : \operatorname{Dom} D \to X$ be closed and D, D^* densely defined. The D is regular if and only if*

$$\mathrm{G}(D) \oplus v\,\mathrm{G}(D^*) \to X \oplus X, \quad ((x, Dx), (-D^*y, y)) \mapsto (x - D^*y, Dx + y),$$

is a unitary isomorphism. In particular $\mathrm{G}(D)$ is complemented and the projection

$$p_D := \begin{pmatrix} (1+D^*D)^{-1} & D^*(1+DD^*)^{-1} \\ D(1+D^*D)^{-1} & DD^*(1+DD^*)^{-1} \end{pmatrix},$$

satisfies $p_D(X \oplus X) = \mathrm{G}(D)$.

Proposition 3.5. *If D is regular, then $\operatorname{Dom} D^*D$ is a core for D.*

Proof. We need to find a sequence $\{(x_n, Dx_n)\}_n$ in $\mathrm{G}(D|_{\operatorname{Dom}(D^*D)})$ with limit (x, Dx). If D is regular, then $1 + \epsilon^2 D^*D$ is surjective as well and $(1+\epsilon^2 D^*D)^{-1}$ map to $\operatorname{Dom}(D^*D)$ for all $\epsilon \in (0,1]$. Then the net

$$\left\{(1+\epsilon^2 D^*D)^{-1}x, D(1+\epsilon^2 D^*D)^{-1}x\right\}_{\epsilon>0}$$

satisfies, as $\epsilon \to 0$,

$$(1+\epsilon^2 D^*D)^{-1}x \to x,$$
$$D(1+\epsilon^2 D^*D)^{-1}x = (1+\epsilon^2 DD^*)^{-1}Dx \to Dx. \qquad \square$$

We now specialise to symmetric operators.

Definition 3.6. The operator D is symmetric if $D \subseteq D^*$. That is, $\operatorname{Dom} D = \operatorname{Dom} D^*$ and $D^*|_{\operatorname{Dom} D} = D$. Equivalently: $\langle Dx, y\rangle = \langle x, Dy\rangle$ for all $x, y \in \operatorname{Dom} D$.

Definition 3.7. A closed, symmetric operator is *self-adjoint* if $\operatorname{Dom} D^* = \operatorname{Dom} D$.

Proposition 3.8. *Let D be a closed, symmetric operator on a Hilbert B-module X. The following are equivalent:*

1. *D is self-adjoint and regular.*
2. *$D \pm \mathrm{i}\colon \operatorname{Dom} D \supseteq X \to X$ have dense range.*
3. *$D \pm \mathrm{i}\colon \operatorname{Dom} D \supseteq X \to X$ are bijective and $(D \pm \mathrm{i})^{-1} \in \operatorname{End}^*_B(X)$.*

Proof. C.f. [Lan95, Chapter 9]. $\square$

Lastly we point out that D is regular if and only if the operator

$$\begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix} : \operatorname{Dom} D^* \oplus \operatorname{Dom} D \to X \oplus X,$$

is self-adjoint and regular. From now we focus on self-adjoint operators.

Finally, there is a very useful tool to detect the regularity of Hilbert module operators called the *local–global principle*. It was first proved by Pierrot and later independently by Kaad and Lesch. We first formulate the local global principle for complemented submodules.

Let $M \subset X$ be a closed submodule, $\pi : B \to \mathbb{B}(H_\pi)$ a representation of B on the Hilbert space H_π and write $X_\pi := X \otimes_B H_\pi$. There is a representation

$$\widehat{\pi} : \operatorname{End}^*_B(X) \to \mathbb{B}(X_\pi), \quad T \mapsto T \otimes 1. \tag{7}$$

Write $M_\pi := M \otimes_B H_\pi \subset X \otimes_B H_\pi$, then M_π is a closed subspace of the Hilbert space X_π.

Theorem 3.9 (Local-global principle for complemented submodules [Pie06])**.** *Let $M \subset X$ be a closed submodule. Then M is complemented if and only if for every irreducible representation $\pi : B \to \mathbb{B}(H_\pi)$ there is an equality $(M_\pi)^\perp = (M^\perp)_\pi$.*

This principle can be used to detect regularity of operators on a Hilbert module via its Hilbert space localisations. This can be an advantage as in Hilbert space we have more tools available to calculate the adjoint of an operator.

Theorem 3.10 (Local–global principle, [Pie06, Théorème 1.18], [KL12, Theorem 4.2])**.** *A closed, symmetric operator D on a Hilbert B-module E is self-adjoint and regular, iff for every irreducible representation $\pi\colon B \to \mathbb{B}(H_\pi)$ of B on a Hilbert space H_π, the operator $D \otimes 1$ on $E \otimes_B H_\pi$ is self-adjoint.*

3.2 The bounded transform

A self-adjoint regular operator is determined by its resolvents $(D \pm \mathrm{i})^{-1}$. These resolvents generate a representation of the C^*-algebra $C_0(\mathbb{R})$ on the module X: the functions $(x \pm \mathrm{i})^{-1}$ generate $C_0(\mathbb{R})$ and the mapping

$$(x \pm \mathrm{i})^{-1} \mapsto (D \pm \mathrm{i})^{-1},$$

determines a $*$-homomorphism $C_0(\mathbb{R}) \to \operatorname{End}^*_B(X)$ written $f \mapsto f(D)$. This is the C_0-*functional calculus of* D. The functional calculus extends to functions $f \in C_b(\mathbb{R}) = M(C_0(\mathbb{R}))$.

Lemma 3.11. *Let $T : \operatorname{Dom} T \to X$ be a densely defined bounded operator such that $T^* : \operatorname{Dom} T^* \to X$ is densely defined and denote by $\overline{T} : X \to X$ the extension of T to all of X. Then T^* is bounded, $\overline{T}, \overline{T^*} \in \operatorname{End}^*_B(X)$ and $\overline{T}^* = \overline{T^*}$.*

Proof. We first prove that T^* is bounded:

$$\|T^* x\| = \sup_{\|y\| \leq 1} \|\langle T^* x, y\rangle\| = \sup_{\|y\|\leq 1, y \in \operatorname{Dom} T} \|\langle x, Ty\rangle\| \leq \|x\| \|T\|.$$

Now the identity $\langle Tx, y\rangle = \langle x, T^* y\rangle$ holds for $x \in \operatorname{Dom} T$ and $y \in \operatorname{Dom} T^*$. It extends to all of X since both operators are bounded. □

Lemma 3.12. *Let $D : \operatorname{Dom} D \to X$ be self-adjoint and regular. Then $(1 + D^2)^{-1/2}$ maps X bijectively onto* $\operatorname{Dom} D$.

Proof. The operator $r := (1 + D^2)^{-1/2}$ is defined through functional calculus. Therefore $r^2 = (1 + D^2)^{-1}$ and thus r has dense range and r^2 maps X onto $\operatorname{Dom} D^2$ which is a core for D. The operators Dr and rD are densely defined on $\operatorname{Ran}(1 + D^2)^{-1/2}$ and $\operatorname{Dom} D$ respectively. A straightforward calculation shows that Dr and rD are mutually adjoint on these domains. Moreover rD is bounded since for $x \in \operatorname{Dom} D$ we have

$$\langle rDx, rDx\rangle = \langle Dx, (1 + D^2)^{-1}Dx\rangle = \langle x, D^2(1 + D^2)^{-1}x\rangle,$$

and $D^2(1 + D^2)^{-1}$ is a bounded operator. Therefore, by Lemma 3.11, $Dr = D(1 + D^2)^{-1/2}$ extends to a bounded operator. Now $(1 + D^2)^{-1/2}$ maps the dense submodule $\operatorname{Ran} r$ onto $\operatorname{Dom} D^2$. Let $x \in X$ be arbitrary and x_n a sequence such that $rx_n \to x$. Then $r^2x_n \to rx$ and $r^2x_n \in \operatorname{Dom} D^2$. Now $D(r^2x_n) = Dr(rx_n)$ converges since Dr is bounded, and since D is closed we find $rx \in \operatorname{Dom} D$ as claimed. □

Definition 3.13. Let $D : \operatorname{Dom} D \to X$ be self-adjoint and regular. Define

$$F_D := D(1 + D^2)^{-1/2}$$

and call it the *bounded transform* of D.

The bounded transform can be viewed as the operator $f(D)$ defined through functional calculus with the bounded function $x \mapsto x(1+x^2)^{-1/2}$. Lemma 3.12 shows that F_D is everywhere equal to the composition of $(1+D^2)^{-1/2}$ with D. This fact is relevant when performing algebraic manipulations with F_D based on properties of D.

3.3 Unbounded Kasparov modules

We now come to the unbounded picture of KK-theory. Instead of considering bounded operators that satisfy certain relations up to compact operators, we now consider unbounded operators that satisfy similar relations up to bounded operators. In practice, proving that an operator is bounded is easier than proving that it is compact. On the other hand, working with unbounded operators comes with its own set of subtleties.

Definition 3.14. Let A and B be C*-algebras. An *unbounded Kasparov* (A, B)*-module* is a triple $(\mathcal{A}, X, D)$, where:

- $\mathcal{A} \subseteq A$ is a dense *-subalgebra.
- X is a $\mathbb{Z}/2$-graded (A, B)-bimodule.
- $D\colon \operatorname{Dom} D \supseteq X \to X$ is an odd, self-adjoint and regular operator.

such that:

- $a(D + \mathrm{i})^{-1} \in \mathbb{K}_B(X)$ for all $a \in \mathcal{A}$.
- For every $a \in \mathcal{A}$, a maps $\operatorname{Dom} D$ into $\operatorname{Dom} D$, and $[D, a]$ extends to an element of $\operatorname{End}^*_B(X)$.

We denote the set of all unbounded Kasparov A, B-modules by $\Psi(A, B)$.

Example 3.15. A spectral triple $(\mathcal{A}, H, D)$ is just an unbounded Kasparov $(A, \mathbb{C})$-module, where A is the closure of $\mathcal{A}$ in $\mathbb{B}(H)$.

Lemma 3.16. *Let* $(\mathcal{A}, X, D) \in \Psi(A, B)$*. For all* $\lambda > 0$ *the operators*

$$(1 + \lambda + D^2)^{-1},\ D(1 + \lambda + D^2)^{-1},\ D^2(1 + \lambda + D^2)^{-1} \in \operatorname{End}^*_B(X),$$

and for all $a \in A$ we have $(1+\lambda+D^2)^{-1}a, D(1+\lambda+D^2)^{-1}a \in \mathbb{K}_B(X)$. Moreover we have the mapping properties

$$(1+\lambda+D^2)^{-1} : X \to \operatorname{Dom} D^2,\ \ D(1+\lambda+D^2)^{-1} : X \to \operatorname{Dom} D,$$

and the norm estimates

$$\|(1+\lambda+D^2)^{-1}\| \le \frac{1}{\lambda},\ \ \|D(1+\lambda+D^2)^{-1}\| \le \frac{1}{\sqrt{\lambda}},\ \ \|D^2(1+\lambda+D^2)^{-1}\| \le 1.$$

Proof. The first statement, as well as the norm estimates, follow from functional calculus. The domain mapping properties follow, for instance, from the factorisation

$$(1+\lambda+D^2)^{-1} = (1+D^2)^{-1}(1+D^2)(1+\lambda+D^2)^{-1},$$

the fact that the function $x \mapsto \frac{1+x^2}{1+\lambda+x^2}$ is bounded and that $(1+D^2)^{-1}$ maps into $\operatorname{Dom} D^2$. The second mapping property follows since $D : \operatorname{Dom} D^2 \to \operatorname{Dom} D$. □

Theorem 3.17 ([BJ83, Proposition 2.2])**.** *If $(\mathcal{A}, X, D) \in \Psi(A,B)$. Then $(X, F_D) \in \mathbb{E}(A,B)$.*

Proof. For simplicity, first assume that A is unital. We need to check that $F_D = D(1+D^2)^{-1/2}$ satisfies (5). First we observe that $F_D^* = F_D$ because D is self-adjoint. Also

$$1 - F_D^2 = 1 - D^2(1-D^2)^{-1} = (1-D^2)^{-1} \in \mathbb{K}_B(X).$$

The non-trivial part is to show that $[F_D, a]$ is compact. We have

$$[F_D, a] = [D(1+D^2)^{-1/2}, a] = [D,a](1+D^2)^{-1/2} + D[(1+D^2)^{-1/2}, a],$$

and this equality holds everywhere since $\operatorname{Ran}(1+D^2)^{-1/2} = \operatorname{Dom} D$ and $a \operatorname{Dom} D \subset \operatorname{Dom} D$.

The first term $[D,a](1+D^2)^{-1/2}$ is compact because $[D,a]$ is bounded and $(1+D^2)^{-1/2}$ is compact.

For the second term: we will use the following integral expression (c.f. [BJ83, Démonstration de Proposition 2.2])

$$(1+D^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda+D^2)^{-1}\mathrm{d}\lambda, \tag{8}$$

which is absolutely norm-convergent integral. We also make use of the algebraic identity

$$[a^{-1}, b] = -a^{-1}[a,b]a^{-1}, \tag{9}$$

which holds whenever both sides of the equation are defined. Now

$$\begin{aligned} D[(1+D^2)^{-1/2}, a] &\overset{(8)}{=} \frac{D}{\pi}\int_0^\infty \lambda^{-1/2}[(1+\lambda+D^2)^{-1}, a]\mathrm{d}\lambda \\ &\overset{(9)}{=} -\frac{D}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda+D^2)^{-1}[1+\lambda+D^2, a](1+\lambda+D^2)^{-1}\mathrm{d}\lambda \\ &= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+\lambda+D^2)^{-1}[D^2,a](1+\lambda+D^2)^{-1}\mathrm{d}\lambda \\ &= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+\lambda+D^2)^{-1}D[D,a](1+\lambda+D^2)^{-1}\mathrm{d}\lambda \\ &\quad -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+\lambda+D^2)^{-1}[D,a]D(1+\lambda+D^2)^{-1}\mathrm{d}\lambda, \end{aligned}$$

and since $[D,a]$ is bounded, for $0 < \lambda < \infty$ the integrand is a compact operator. Thus, to conclude that $D[(1+D^2)^{-1/2}, a]$ is compact, it suffices to prove that the integral is norm convergent.

Since $[D, a]$ is bounded by assumption, and $D(1 + \lambda + D^2)^{-1}D$ is contractive by functional calculus: we have

$$\begin{aligned}\|D[(1 + D^2)^{-1/2}, a]\| &\leq \frac{2}{\pi}\|[D, a]\| \int_0^\infty \lambda^{-1/2}(1 + \lambda + D^2)^{-1}\mathrm{d}\lambda \\ &\leq \frac{2}{\pi}\|[D, a]\| \int_0^\infty \lambda^{-1/2}(1 + \lambda)^{-1}\mathrm{d}\lambda.\end{aligned}$$

The integral

$$\int_0^\infty \lambda^{-1/2}(1 + \lambda)^{-1}\mathrm{d}\lambda$$

converges absolutely, because $\lambda^{-1/2}(1 + \lambda)^{-1} \sim \lambda^{-1/2}$ as $\lambda \to 0$, and $\lambda^{-1/2}(1 + \lambda)^{-1} \sim \lambda^{-3/2}$ as $\lambda \to \infty$. This proves that $[F_D, a]$ is bounded. For the nonunital case one considers commutators of the form $[F_D, ab] = a[F_D, b] + [F_D, a]b$ with $a, b \in \mathcal{A}$. Using compactness of $a(1 + D^2)^{-1}$, the above proof then shows that $[F_D, ab]$ is compact. Since $\mathcal{A}$ is dense in A, $\mathcal{A}^2$ is dense in A^2, since A is a C^*-algebra, A^2 is dense in A. Therefore $[F_D, a]$ is compact for all $a \in A$. □

Theorem 3.18 ([BJ83, Proposition 2.3])**.** *Every class in* $\mathrm{KK}(A, B)$ *can be represented by an element in* $\Psi(A, B)$*. Namely, the map*

$$\Psi(A, B) \to \mathrm{KK}(A, B), \qquad (\mathcal{A}, X, D) \mapsto [X, F_D]$$

is surjective.

Remark 3.19. Note that this theorem does not state that every bounded Kasparov module in $\mathbb{E}(A, B)$ is the bounded transform of an element of $\Psi(A, B)$: this is not true. Rather, every bounded Kasparov module lifts to an unbounded module in $\Psi(A, B)$, and the bounded transform of the lift is homotopic to the original module.

3.4 Homotopy of unbounded Kasparov modules

The homotopy relation can be defined for unbounded Kasparov modules, along the same lines as that for bounded Kasparov modules. There is one subtlety that needs to be taken into account, which has to do with the dense subalgebra $\mathcal{A} \subset A$. This algebra can vary when we choose representatives for $KK_*(A, B)$. This creates problems when defining direct sums of unbounded Kasparov modules, and also creates technical difficulties when working with the natural notion of homotopy. Given an self-adjoint regular operator D on a Hilbert C^*-module X, define

$$\begin{aligned}\mathrm{Lip}(D) &:= \left\{T \in \mathrm{End}^*_B(X) : T : \mathrm{Dom}\, D \to \mathrm{Dom}\, D,\ [D, T] \in \mathrm{End}^*_B(X)\right\}, \\ \mathrm{Lip}^0(D) &:= \left\{T \in \mathrm{Lip}(D) : T(D \pm \mathrm{i})^{-1}, (D \pm \mathrm{i})^{-1}T \in \mathbb{K}_B(X)\right\},\end{aligned}$$

We note that $\mathrm{Lip}^0(D)$ and $\mathrm{Lip}(D)$ are $*$-algebras. Using these $*$-algebras, we can formulate a more flexible notion of cycle for $KK_*(A, B)$, that makes reference only to the C^*-algebra A.

Definition 3.20 ([DM20])**.** Let (A, B) be a pair of C^*-algebras. An *unbounded cycle* for (A, B) is a pair (X, D) where:

- X is a $\mathbb{Z}/2$-graded (A, B)-bimodule.
- $D\colon\ \mathrm{Dom}\, D \supseteq X \to X$ is an odd, self-adjoint and regular operator.

such that:

- $a(D + \mathrm{i})^{-1} \in \mathbb{K}_B(X)$ for all $a \in A$.
- $A \subset \overline{\mathrm{Lip}^0(D)}$

We enlarge $\Psi(A, B)$ accordingly. With minor modifications, it can be shown that the bounded transform of an unbounded cycle is a bounded Kasparov module. We thus retain the surjective map $\Psi(A, B) \to KK(A, B)$. The main advantage of unbounded cycles is that their direct sum

$$(X^0, D^0) \oplus (X^1, D^1) := (X^0 \oplus X^1, D^0 \oplus D^1),$$

is well defined. This is not the case for unbounded Kasparov modules $(\mathcal{A}^i, X^i, D^i)$, $i = 0, 1$, unless the intersection $\mathcal{A}^0 \cap \mathcal{A}^1$ is dense in A. In the context of homotopies, unbounded cycles add a comparable flexibility, since we do not to worry about variations in the dense subalgebra $\mathcal{A} \subset A$.

Definition 3.21. Let (X^0, D^0) and (X^1, D^1) be even unbounded cycles for (A, B). We say they are *homotopic*, if there exists an even unbounded $(A, B[0, 1])$-cycle (X, D), such that

$$(X \otimes_{\mathrm{ev}_0} B, D \otimes 1) \simeq (X^0, D^0) \quad \text{and} \quad (X \otimes_{\mathrm{ev}_1} B, D \otimes 1) \simeq (X^1, D^1).$$

Here we use

$$\mathrm{ev}_t \colon B[0, 1] \to B, \qquad f \mapsto f(t),$$

to equip B with a $(B[0, 1], B)$-bimodule structure.

This relation enjoys the same properties as the bounded homotopy relation.

Proposition 3.22 ([Kaa20, DM20])**.** *Homotopy of unbounded cycles is an equivalence relation on* $\Psi(A, B)$.

The main result regarding the equivalence relation is that we obtain the same KK-group as in the bounded picture.

Theorem 3.23 ([Kaa20, DM20])**.** *The Kasparov group* $\mathrm{KK}(A, B)$ *equals the set of homotopy equivalence classes of* $\Psi(A, B)$.

The unbounded homotopy relation allows for more flexibility than the bounded relation, as can be seen in the following example.

Example 3.24. For $\lambda > 0$ notice that

$$D(\lambda^2 + D)^{-1/2} = \frac{D}{\lambda} \left(1 + \left(\frac{D}{\lambda} \right)^2 \right)^{-1/2}.$$

The scaling $D \mapsto \lambda D$ actually provides a homotopy of unbounded modules. It is clear that this homotopy does not exist in the bounded picture.

Lastly we mention that another, more algebraic equivalence relation can be defined on $\Psi(A, B)$. The relation is called *bordism* [DGM18], and uses another generalisation of unbounded Kasparov modules, allowing for the use of symmetric, non-self-adjoint operators on Hilbert C^*-modules [Hil10].

4 The Kasparov product

For all separable C*-algebras A, B, C and D, Kasparov constructed two associative bilinear pairings

$$\begin{aligned} \otimes \colon \; & \mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \to \mathrm{KK}(A, C), \\ \boxtimes \colon \; & \mathrm{KK}(A, B) \times \mathrm{KK}(C, D) \to \mathrm{KK}(A \otimes C, B \otimes D). \end{aligned}$$

The first one is called the *(internal) Kasparov product* and the second one is called the *external Kasparov product*. The seminal work of Kasparov consisted in showing that both products exist uniquely up to homotopy of (bounded) Kasparov modules. However, it is in general technically difficult to give an explicit general construction for these products.

Since Baaj and Julg showed that KK-theory can be represented by unbounded modules in [BJ83], it has taken a considerable amount of time for the internal Kasparov product to be better understood. It turns out that in desirable situations, simple formulae are available. The cost is the extensive use of unbounded operators on Hilbert C*-modules.

4.1 Existence of the Kasparov product

Usually, the approach to finding the Kasparov product of two given (bounded) Kasparov modules is through a "guess and check" process. More precisely, we have the following theorem due to Connes and Skandalis:

Theorem 4.1 ([CS84, Appendix A])**.** *Let $(X, F_1) \in \mathbb{E}(A, B)$ and $(Y, F_2) \in \mathbb{E}(B, C)$. The Kasparov product $(X \otimes_B Y, F) \in \mathbb{E}(A, C)$ is, uniquely up to homotopy, characterised by the following properties:*

Connection condition *For all $x \in X$: the operator*

$$[y \mapsto F(x \otimes y) - \gamma(x) \otimes F_2 y] \in \mathbb{K}(Y, X \otimes_B Y),$$

where γ is the grading on X.

Positivity condition *There exists $0 < \kappa < 2$, such that for all $a \in A$:*

$$a^*[F_1 \otimes 1, F]a \geq -\kappa a^* a \mod \mathbb{K}(X \otimes_B Y).$$

Can we get our hands on this operator F? We would like to think of F as being of the form $F_1 \otimes 1 + \gamma \otimes F_2$ acting on $X \otimes_B Y$. There is a simple algebraic obstruction to this naïve idea: notice that

$$(\gamma \otimes F_2)(xb \otimes y) = \gamma(xb) \otimes F_2 y = \gamma(x) \otimes bF_2 y$$

but

$$(\gamma \otimes F_2)(x \otimes by) = \gamma(x) \otimes F_2 by$$

which are not the same unless $[F_2, B] = 0$. Moreover, even if $\gamma \otimes F_2$ is well-defined, the sum $F_1 \otimes 1 + \gamma \otimes F_2$ does generally not satisfy the conditions 5 for a bounded Kasparov module.

In order to make sense of $1 \otimes F_2$, a partial solution is to use the stabilisation theorem and associated frames (Section 2.4).

Let X be a countably generated Hilbert B-module and $V : X \to H \otimes B$ be the isometry coming from Kasparov's stablisation theorem. Extend V to $V : X \to H \otimes B^+$ if B is non-unital. Let $\{e_i\}_{i \in \mathbb{N}}$ be a countable basis of H. Define the frame $\{x_i\}_{i \in \mathbb{N}}$ of X:

$$x_i := V^*(e_i \otimes 1)$$

Then

$$x = \sum_i x_i \langle x_i, x \rangle, \qquad \text{for all } x \in X.$$

The isometry V extends to a map $V \otimes 1 : X \otimes_B Y \to H \otimes B^+ \otimes_B Y \simeq H \otimes Y$. Notice that in the tensor product module $H \otimes Y$ there is no balancing over the C*-algebra B anymore. So we can set

$$\hat{F}_2 : X \otimes_B Y \to X \otimes_B Y, \qquad \hat{F}_2(x \otimes y) := (V^* \otimes 1)(1 \otimes F_2)(V \otimes 1)(\gamma(x) \otimes y).$$

Proposition 4.2. *$\hat{F}_2$ satisfies the Connection condition.*

Proof. Let $\{e_i\}$ be a basis of H. Then

$$x = \sum_i x_i \langle x_i, x \rangle.$$

We have

$$\begin{aligned}
\hat{F}_2(x \otimes y) - \gamma(x) \otimes F_2 y &= \sum_i x_i \otimes F_2 \langle x_i, \gamma(x) \rangle y - \gamma(x) \otimes F_2 y \\
&= \sum_i x_i \otimes F_2 \langle x_i, \gamma(x) \rangle y - \sum_i x_i \langle x_i, \gamma(x) \rangle \otimes F_2 y \\
&= \sum_i x_i \otimes [F_2, \langle x_i, \gamma(x) \rangle] y.
\end{aligned}$$

The operators $[F_2, \langle x_i, \gamma(x) \rangle] \in \mathbb{K}_C(X)$ because $(Y, F_2) \in \mathbb{E}(B, C)$. Therefore, the map $y \mapsto x_i \otimes [F_2, \langle x_i, \gamma(x) \rangle] y$ is also compact.This implies that for any finite partial sum: the operator

$$y \mapsto \sum_{i=0}^{N} x_i \otimes [F_2, \langle x_i, \gamma(x) \rangle] y$$

is compact.

It suffices to check that this series converges in norm as $N \to \infty$. Then it is the norm limit of a sequence of compact operators, hence compact. To this end, notice that

$$\begin{aligned}
\sup_{\|y\| \leq 1} \left\| \sum_{i=N+1}^{M} x_i \otimes [F_2, \langle x_i, \gamma(x) \rangle y] \right\| &\leq \sup_{\|y\| \leq 1} \|x\| \left\| \begin{pmatrix} [F_2, \langle x_{N+1}, \gamma(x) \rangle] \\ [F_2, \langle x_{N+2}, \gamma(x) \rangle] \\ \vdots \\ [F_2, \langle x_M, \gamma(x) \rangle] \end{pmatrix} \right\| \|y\| \\
&\leq \sup_{\|y\| \leq 1} 2\|x\| \|F_2\| \left\| \begin{pmatrix} \langle x_{N+1}, \gamma(x) \rangle \\ \langle x_{N+2}, \gamma(x) \rangle \\ \vdots \\ \langle x_M, \gamma(x) \rangle \end{pmatrix} \right\| \|y\|.
\end{aligned}$$

But since $\sum_i x_i \langle x_i, \gamma(x) \rangle$ converges to $\gamma(x)$ in norm, we have

$$\left\| \begin{pmatrix} \langle x_{N+1}, \gamma(x) \rangle \\ \langle x_{N+2}, \gamma(x) \rangle \\ \vdots \\ \langle x_M, \gamma(x) \rangle \end{pmatrix} \right\| \to 0$$

and hence the tail converges to 0 in norm. □

Now we define the operator $G = F_1 \otimes 1 + \hat{F}_2$ and wish to show that it represents the Kasparov product of (X, F_1) and (Y, F_2). This is unfortunately not true as $(X \otimes_B Y, G)$ fails to be a Kasparov module in general. Moreover, neither the Connection condition nor the Positivity condition holds.

1. Connection condition: we have

$$G(x \otimes y) - \gamma(x) \otimes F_2 y = (F_1 x \otimes y) + (\hat{F}_2(x \otimes y) - \gamma(x) \otimes F_2 y).$$

So the connection condition holds iff the operator $y \mapsto F_1 x \otimes y$ is compact, which is not true in general.

2. Positivity condition: $[F_1 \otimes 1, G] = [F_1 \otimes 1, F_1 \otimes 1] + [F_1 \otimes 1, \hat{F}_2]$. We have claimed that $[F_1 \otimes 1, F_1 \otimes 1]$ is positive as desired, but for the second term $[F_1 \otimes 1, \hat{F}_2]$ there is no guarantee at all.
3. Conditions for a Kasparov module: We have $G - G^* = 0 \in \mathbb{K}_B(X)$. However, for the other conditions:
   - $[G, a] = [F_1, a] \otimes 1 + [\hat{F}_2, a]$, and neither of the summands is compact in general.
   - $G^2 - 1 = F_1^2 \otimes 1 + \hat{F}_2^2 + [F_1 \otimes 1, \hat{F}_2] - 1$. Unfortunately, we know nothing about $[F_1 \otimes 1, \hat{F}_2]$.

Kasparov provided a solution to this issue. It relies on a deep result in the analysis of C^*-algebras. That is, one replaces $F_1 \otimes 1 + \hat{F}_2$ by another operator $M^{1/2}(F_1 \otimes 1) + N^{1/2}\hat{F}_2$, for some carefully chosen operators M and N. In the specific context of the Kasparov product, the existence of the operators M and N is guatanteed by the following theorem.

Theorem 4.3 (Kasparov's technical theorem)**.** *There exist even operators $M, N \in \mathrm{End}^*_C(X \otimes_B Y)$ with $M + N = 1$, satisfying:*

1. $M(\mathbb{K}(X) \otimes 1) \subseteq \mathbb{K}(X \otimes_B Y)$.
2. $N(\hat{F}_2^2 - 1) \in \mathbb{K}(X \otimes_B Y)$, $N[\hat{F}_2, a] \in \mathbb{K}(X \otimes_B Y)$ *and* $N[F_1 \otimes 1, \hat{F}_2] \in \mathbb{K}(X \otimes_B Y)$.
3. $[F_1 \otimes 1, N] \in \mathbb{K}(X \otimes_B Y)$, $[\hat{F}_2, N] \in \mathbb{K}(X \otimes_B Y)$ *and* $[N, a] \in \mathbb{K}(X \otimes_B Y)$.

Then $M^{1/2}$ and $N^{1/2}$ satisfy 1–3 as well, and

$$(X \otimes_B Y, M^{1/2}(F_1 \otimes 1) + N^{1/2}\hat{F}_2) \in \mathbb{E}(A, C)$$

is the Kasparov product of (X, F_1) and (Y, F_2).

Here we emphasize once more that the most important aspect of the above result is that $M^{1/2}(F_1 \otimes 1) + N^{1/2}\hat{F}_2)$ is a Kasparov module. This covers the existence of the following cornerstone result in KK-theory.

Theorem 4.4. *The Kasparov product exists and is unique up to homotopy.*

Proof. This is established through the following process:

1. Use *Kasparov's stablisation theorem* to find $\hat{F}_2$.
2. Use *Kasparov's technical theorem* to find M and N and to build a Kasparov module.
3. Use *Connes–Skandalis' theorem* to prove the existence and uniqueness. □

4.2 The unbounded Kasparov product

The difficulty in constructing Kasparov products can be seen from examples in geometry. A Kasparov module (X, F) is modelled on a zeroth-order pseudodifferential operator F, coming from the bounded transform of a first-order unbounded differential operator D. In many cases, e.g. Riemannian submersions between spin^c-manifolds (c.f. Section 5.3), the Kasparov product of the geometric differential operators involved — a "horizontal" operator and a "vertical" operator — is naturally represented by the bounded transform F of the sum of these operators, acting on the tensor product Hilbert C*-module. The bounded transform destroys the linearity of this operation and makes the operator F quite involved.

It is better to investigate the mysterious operators M and N, in Kasparov's technical lemma for external products. Let $(X, F_1) \in \mathbb{E}(A, B)$ and $(Y, F_2) \in \mathbb{E}(C, D)$. In the search for their external Kasparov products, there is no need to apply Kasparov's stablisation theorem for balancing. But one still needs to apply Kasparov's technical theorem to find operators $M, N \in \mathrm{End}^*_{C \otimes D}(X \otimes Y)$ satisfying similar requirements as Theorem 4.3 (up to replacing balanced tensor products by the unbalanced ones). However, while working with unbounded Kasparov modules, such operators M and N do not appear anymore: they show up precisely due to the bounded transform of unbounded modules.

Theorem 4.5 ([BJ83, Théorème 3.2])**.** *The exterior Kasparov product of* $(X, S) \in \Psi(A, B)$ *and* $(Y, T) \in \Psi(C, D)$ *is represented by*

$$(X \otimes Y, S \otimes 1 + 1 \otimes T) \in \Psi(A \otimes C, B \otimes D).$$

Proof. Set $s := S \otimes 1$ and $t := 1 \otimes T$. Then $s + t$ is defined on $\operatorname{Dom} S \otimes^{\mathrm{alg}} \operatorname{Dom} T$, which is a core for $s + t$. Notice that

$$(st + ts)(x \otimes y) = S\gamma(x) \otimes Ty + \gamma(Sx) \otimes Ty = (S\gamma + \gamma S)x \otimes Ty = 0.$$

So $st + ts = 0$ on $\operatorname{Dom} S \otimes^{\mathrm{alg}} \operatorname{Dom} T$ and hence on $\operatorname{Dom} s \cap \operatorname{Dom} t$.

Therefore, $(s + t)^2 = s^2 + t^2 + st + ts = s^2 + t^2$ on $\operatorname{Dom} s \cap \operatorname{Dom} t$. We have

$$(2 + t^2 + s^2)(1 + s^2)^{-1}(1 + t^2)^{-1} = (1 + s^2)^{-1} + (1 + t^2)^{-1}$$

has dense range. Therefore, $2 + t^2 + s^2$ has dense range and $s + t$ is self-adjoint and regular.

Now consider the bounded transform of the unbounded operator $s + t$. We have

$$\begin{aligned}(s+t)(1+(s+t)^2)^{-1/2} &= s(1+s^2+t^2)^{-1/2} + t(1+s^2+t^2)^{-1/2}\\ &= s(\tfrac{1}{2}+s^2)^{-1/2}\boxed{(\tfrac{1}{2}+s^2)^{1/2}(1+s^2+t^2)^{-1/2}} \longrightarrow M^{1/2}\\ &\quad + t(\tfrac{1}{2}+t^2)^{-1/2}\boxed{(\tfrac{1}{2}+t^2)^{1/2}(1+s^2+t^2)^{-1/2}} \longrightarrow N^{1/2}\end{aligned}$$

The operators M and N defined above satisfy the conditions in Kasparov's technical theorem (Theorem 4.3), and hence represents the external Kasparov product. □

Remark 4.6. By Example 3.24, the operator $s(\frac{1}{2} + s^2)^{-1/2}$ is homotopic to the bounded transform of s.

Now we return to construct the internal Kasparov products of unbounded Kasparov modules. Similar to the bounded cases, this is worked out through a guess-and-check process, using the following Connes–Skandalis type theorem due to Kucerovsky.

Theorem 4.7 ([Kuc97, Theorem 13])**.** *Let* $(\mathcal{A}, X, S) \in \Psi(A, B)$ *and* $(\mathcal{B}, Y, T) \in \Psi(B, C)$. *If* $(\mathcal{A}, X \otimes_B Y, D) \in \Psi(A, C)$ *satisfies the following conditions:*

Connection condition *For all* x *in a dense subspace of* X*: the operator*

$$y \mapsto D(x \otimes y) - \gamma(x) \otimes Ty$$

extends to a bounded (hence adjointable) operator in $\operatorname{End}^*_C(Y, X \otimes_B Y)$.

Domain condition $\operatorname{Dom} D \subseteq \operatorname{Dom}(S \otimes 1)$.

Positivity condition *There exists* $\kappa > 0$ *such that for all* $\xi \in \operatorname{Dom} D$*:*

$$\langle D\xi, (S \otimes 1)\xi\rangle + \langle (S \otimes 1)\xi, D\xi\rangle \geq -\kappa\langle \xi, \xi\rangle.$$

Then $(\mathcal{A}, X \otimes_B Y, D)$ *represents the Kasparov product of* (X, S) *and* (Y, T).

Remark 4.8.

1. The domain condition indicates that we should think of D as an operator

   $$D = S \otimes 1 + \hat{T},$$

   hence $\operatorname{Dom} D = \operatorname{Dom} S \otimes 1 \cap \operatorname{Dom} \hat{T} \subseteq \operatorname{Dom} S \otimes 1$.
2. The positivity and boundedness condition is, more or less, a rephrasal of the positivity condition in the unbounded picture. But we need to care about the domain issue: we do not yet know whether D and S are composable, so the graded commutator may not make sense.

4.2.1 Connections

As with the bounded Kasparov product, we wish to make sense of the operator "$1 \otimes T$" as an unbounded operator acting on $X \otimes_B Y$. The good news here is that we can explicitly describe such operators with the extra input of a *connection*. Connections are central objects in differential geometry. Their algebraic theory has been studied in detail by Cuntz and Quillen [CQ95].

In contrast, connections in unbounded KK-theory involve analytic structure as well. As it turns out, the theory of *operator spaces* provides the right context to for the algebra and analysis to interact fruitfully. Recall that for a C^*-algebra A, the $*$-algebra $M_n(A)$ carries a unique C^*-norm. An operator space is a closed subspace of a C*-algebra, and as such comes equipped with matrix norms as well. Several differential-geometric objects can be interpreted analytically in this context.

Some results concerning operator spaces, operator algebras and operator modules can be found in Section 5.2. For the general theory we refer to [?]. The interplay between operator spaces and unbounded KK-theory were first described in [Mes14], soon thereafter followed by [KL13]. It was further developed in [BKM18].

Definition 4.9. Let $(\mathcal{B}, Y, T) \in \Psi(B, C)$ be an unbounded Kasparov module. The module of noncommutative differential 1-forms for $(\mathcal{B}, Y, T)$ is the operator space

$$\Omega^1_T(\mathcal{B}) := \overline{\operatorname{span}}\{b[T, b'] \mid b, b' \in \mathcal{B}\} \subseteq \operatorname{End}^*_C(Y).$$

The closure is with respect to the norm topology of $\operatorname{End}^*_C(Y)$.

Remark 4.10. In fact $\Omega^1_T(\mathcal{B})$ is a $(\mathcal{B}, \mathcal{B})$-bimodule with left and right module structures given by

$$a \cdot b[T, b'] := ab[T, b'], \quad b[T, b'] \cdot c := b[T, b'c] - bb'[T, c].$$

Note that the Leibniz rule $[T, ab] = a[T, b] + [T, a]b$ implies that the right $\mathcal{B}$-module structure is defined by right operator multiplication.

Definition 4.11. Let X be a Hilbert B-module and $(\mathcal{B}, Y, T) \in \Psi(B, C)$. A $(\mathcal{B}, Y, T)$-connection on X is a densely defined even linear map

$$\nabla\colon X \supseteq \mathcal{X} \to X \otimes^{\mathrm{h}}_B \Omega^1_T(\mathcal{B}) \subseteq X \otimes_B \operatorname{End}^*_C(Y),$$

where $\mathcal{X} \subseteq X$ is a dense subspace, which is the domain of ∇; $\otimes^{\mathrm{h}}$ refers to the *Haagerup tensor product*, such that

$$\nabla(xb) = \nabla(x)b + \gamma(x) \otimes [T, b], \qquad \text{for all } x \in \mathcal{X} \text{ and } b \in \mathcal{B}.$$

Definition 4.12. A $(\mathcal{B}, Y, T)$-connection ∇ is *Hermitian*, if

$$\langle x_1, \nabla x_2 \rangle - \langle \nabla x_1, x_2 \rangle = [T, \langle x_1, x_2 \rangle], \qquad \text{for all } x_1, x_2 \in \operatorname{Dom} \nabla.$$

Remark 4.13. A precise definition of the Haagerup tensor product is given in Section 5.2. We briefly explain why the Haagerup tensor product is useful here:

- The Haagerup tensor product is characterised by the universal property that the multiplication map $B \otimes^{\mathrm{h}}_B B \to B$ is continuous for any C*-algebra B.
- Given any C*-algebra B and Hilbert B-module X, the multiplication map $X \otimes^{\mathrm{h}}_B B \to X$ is continuous.
- Given a Hilbert B-module X and a (B, C)-bimodule Y. There is a completely bounded isomorphism of operator modules $X \otimes_B Y \simeq X \otimes^{\mathrm{h}}_B Y$.

Definition 4.14. Let $(\mathcal{B}, Y, T) \in \Psi(B, C)$. Given a densely defined $(\mathcal{B}, Y, T)$-connection ∇ on X we define the operator

$$1 \otimes_\nabla T \colon X \otimes_B Y \supseteq \mathcal{X} \otimes_{\mathcal{B}}^{\text{alg}} \operatorname{Dom} T \to X \otimes_B Y$$
$$(1 \otimes_\nabla T)(x \otimes y) := \gamma(x) \otimes Ty + \nabla(x)y.$$

Proposition 4.15. *The operator $D := S \otimes 1 + 1 \otimes_\nabla T$ is well-defined and satisfies Kucerovsky's Connection condition. If ∇ is Hermitian, then D is symmetric.*

Proof. For well-definedness we observe

$$\begin{aligned} 1 \otimes_\nabla T(xb \otimes y) &= \gamma(xb) \otimes Ty + \nabla(xb)y \\ &= \gamma(x) \otimes bTy + \nabla(x)by + \gamma(x) \otimes [T, b]y \\ &= \gamma(x) \otimes Tby + \nabla(x)by \\ &= 1 \otimes_\nabla T(x \otimes by). \end{aligned}$$

So the operator $1 \otimes_\nabla T$ is well-defined. Now

$$(S \otimes 1 + 1 \otimes_\nabla T)(x \otimes y) - \gamma(x) \otimes Ty = Sx \otimes y + \nabla(x)y.$$

For $x \in \operatorname{Dom} S$ it is clear that $y \mapsto Sx \otimes y$ is bounded, and $y \mapsto \nabla(x)y$ is bounded because $\nabla(\gamma(x)) \in X \otimes_B \operatorname{End}_C^*(Y)$, proving that D satisfies the connection condition. Since $S \otimes 1$ is self-adjoint, for D to be symmetric it suffices to prove that $1 \otimes_\nabla T$ is symmetric. Since T is self-adjoint and ∇ is Hermitian we have

$$\begin{aligned} \langle (1 \otimes_\nabla T)(x_1 \otimes y_1), x_2 \otimes y_2 \rangle &= \langle \gamma(x_1) \otimes Ty_1, x_2 \otimes y_2 \rangle + \langle \nabla(x_1)y_1, x_2 \otimes y_2 \rangle \\ &= \langle Ty_1, \langle \gamma(x_1), x_2 \rangle y_2 \rangle + \langle y_1, \langle \nabla(x_1), x_2 \rangle y_2 \rangle \\ &= \langle y_1, T\langle \gamma(x_1), x_2 \rangle y_2 \rangle + \langle y_1, \langle \nabla(x_1), x_2 \rangle y_2 \rangle \\ &= \langle y_1, \langle \gamma(x_1), x_2 \rangle Ty_2 \rangle + \langle y_1, \langle x_1, \nabla(x_2) \rangle y_2 \rangle \\ &= \langle y_1, \langle x_1, \gamma(x_2) \rangle Ty_2 \rangle + \langle y_1, \langle x_1, \nabla(x_2) \rangle y_2 \rangle \\ &= \langle x_1 \otimes y_1, (1 \otimes_\nabla T)(x_2 \otimes y_2) \rangle, \end{aligned}$$

completing the proof that $1 \otimes_\nabla T$ is symmetric. □

4.2.2 The constructive Kasparov product

Let $(\mathcal{A}, X, S) \in \Psi(A, B)$ and $(\mathcal{B}, Y, T) \in \Psi(B, C)$. Suppose that ∇ is a $(\mathcal{B}, Y, T)$-connection on X. Then the operator

$$D := S \otimes 1 + 1 \otimes_\nabla T$$

satisfies the Kucerovsky's Connection condition, making good sense of the operator "$1 \otimes T$" on $X \otimes_B Y$.

Lemma 4.16. *Let $(\mathcal{A}, X, S)$ and $\mathcal{B}, Y, T)$ be unbounded Kasparov modules and $\nabla : \mathcal{X} \to X \otimes_B^{\text{h}} \Omega_T^1(\mathcal{B})$ a Hermitian connection. Suppose that $D := S \otimes 1 + 1 \otimes_\nabla T$ is self-adjoint and regular on $\operatorname{Dom} S \otimes 1 \cap \operatorname{Dom} \gamma \otimes_\nabla T$. Then $a(D+i)^{-1}$ is compact.*

Proof. There is a map $\mathbb{K}(X) \to \operatorname{End}_C^*(X \otimes_B Y)$ defined by $K \mapsto K \otimes 1$. For $x \in X$ define $T_x : B \to X$ by $T_x(b) := xb$, so that $T_x^* : X \to B$ is given by $T_x^*(x_0) = \langle x, x_0 \rangle$. Similarly we define $T_x : Y \to X \otimes_B Y$ by $T_x(y) := x \otimes y$. The rank one operator $T_{x_1, x_2} : X \to X$ is equal to $T_{x_1} T_{x_2}^*$, and the connection condition says that $DT_x - T_x T$ extends to a bounded operator. Then $T_x^* D - TT_x^*$ is bounded as well. Now for $x_1, x_2 \in \operatorname{Dom} s \cap \operatorname{Dom} t$ and $b \in B$ we can write

$$\begin{aligned} T_{x_1 b} T_{x_2}^* (D+i)^{-1} &= T_{x_1} b(T+i)^{-1} T_{x_2}^* - T_{x_1}(b(T+i)^{-1} T_{x_2}^* - T_{x_2}^*(D+i)^{-1}) \\ &= T_{x_1} b(T+i)^{-1} T_{x_2}^* - T_{x_1} b(T+i)^{-1}(T_{x_2}^* D - TT_{x_2}^*)(D+i)^{-1}, \end{aligned}$$

and since $b(T+i)^{-1} \in \mathbb{K}(Y)$ and $(T^*_{x_2}D - TT^*_{x_2}$ is bounded adjointable, we deduce that $T_{x_1 b}T^*_{x_2}(D+i)^{-1}$ is compact. Now the rank one operators $T_{x_1 b}T^*_{x_2}$ span a dense subspace of $\mathbb{K}(X)$, and we conclude that for all $K \in \mathbb{K}(X)$ the operator $K(D+i)^{-1}$ is compact on $X \otimes_B Y$. Now let u_n be an approximate unit for $\mathbb{K}(X)$ and observe that then also $u_n a \in \mathbb{K}(X)$ and thus $u_n a(D+i)^{-1} \in \mathbb{K}(X \otimes_B Y)$. Since also

$$u_n a(D+i)^{-1} = u_n a(S+i)^{-1}(S+i)(D+i)^{-1}$$

and $(S+i)(D+i)^{-1} \in \operatorname{End}^*_C(X \otimes_B Y)$ and $a(S+i)^{-1} \in \mathbb{K}(X)$, the sequence is norm-convergent, so $a(D+i)^{-1} \in \mathbb{K}(X \otimes_B Y)$. □

However, showing that $(\mathcal{A}, X \otimes_B Y, S \otimes 1 + 1 \otimes_\nabla T)$ represents the Kasparov product requires more work. First we require that it is an unbounded Kasparov module, which means we need to show that:

- $S \otimes 1 + 1 \otimes_\nabla T$ is self-adjoint and regular.
- $[1 \otimes_\nabla T, a] := \nabla a - (a \otimes 1)\nabla$ is bounded.

We also require that Kucerovsky's conditions to be satisfied. Though the Connection condition is true by the construction of connections, the Domain condition and Positivity condition might fail. Unpacking Kucerovsky's theorem in the case of a sum $D = S \otimes 1 + \gamma \otimes_\nabla T$ yields the following reformulation.

Theorem 4.17. *Let $(\mathcal{A}, X, S) \in \Psi(A, B)$ and $(\mathcal{B}, Y, T) \in \Psi(B, C)$. Let $\nabla \colon X \supseteq \mathcal{X} \to X \otimes^{\mathrm{h}}_{\mathcal{B}} \Omega^1_T(\mathcal{B})$ be a $(\mathcal{B}, Y, T)$-connection. If:*

- *$S \otimes 1 + 1 \otimes_\nabla T$ is essentially self-adjoint and regular.*
- *There exists $\lambda > 0$, such that $\langle (S \otimes 1)\xi, (1 \otimes_\nabla T)\xi \rangle + \langle (1 \otimes_\nabla T)\xi, (S \otimes 1)\xi \rangle \geq -\lambda \langle \xi, \xi \rangle$ for all $\xi \in \operatorname{Dom}(S \otimes 1 + 1 \otimes_\nabla T)$.*
- *$[1 \otimes_\nabla T, a]$ is bounded for every $a \in \mathcal{A}$.*

Then $(\mathcal{A}, X \otimes_B Y, S \otimes 1 + 1 \otimes_\nabla T) \in \Psi(A, C)$ and represents the Kasparov product of $(\mathcal{A}, X, S)$ and $(\mathcal{B}, Y, T)$.

In most examples the boundedness of $[\gamma \otimes_\nabla T, a]$ is the main obstacle and violated in various situations, e.g. non-isometric actions on Riemannian manifolds (Section 5.4) and the Hopf fibration of quantum SU(2).

The positivity condition as stated here may also fail, in particular in noncompact and non-unital situations. Recent research has gone into finding weaker versions of this condition [KVS19, Dun23].

A sufficient condition for self-adjointness and regularity of $S \otimes 1 + 1 \otimes_\nabla T$ consists in a certain "smallness" of the graded commutator $[S \otimes 1, 1 \otimes_\nabla T]$. This condition then automatically implies the positivity condition. In particular, a *weakly anti-commuting pair* of operators gives rise to self-adjoint and regular sum, c.f. [Mes14, KL12, LM19]. We give the most general formulation of this result.

Theorem 4.18 ([LM19, Theorem 2.6])**.** *Let (s, t) be a pair of self-adjoint regular operators. Set*

$$\begin{aligned} \mathcal{F}(s,t) &:= \operatorname{Dom} st \cap \operatorname{Dom} ts \\ &= \{ e \in \operatorname{Dom} s \cap \operatorname{Dom} t \mid se \in \operatorname{Dom} t, te \in \operatorname{Dom} s \}. \end{aligned}$$

Suppose that:

- *There is a core $\mathcal{E} \subseteq \operatorname{Dom} t$ for t, such that the resolvent*

$$(s + \mathrm{i})^{-1}\mathcal{E} \subseteq \mathcal{F}(S, t).$$

- *There exists $C > 0$, such that for all $e \in \mathcal{F}(s,t)$:*

$$\langle [s,t]e, [s,t]e \rangle \leq C(\langle se, se \rangle + \langle te, te \rangle + \langle e, e \rangle).$$

Then $s + t$ is self-adjoint and regular.

If $(S \otimes 1, \gamma \otimes_\nabla T)$ form a weakly anti-commuting pair, then their sum is self-adjoint and regular, has compact resolvent, and the positivity condition is satisfied. In such cases, the commutators $[\gamma \otimes_\nabla T, a]$ can still be unbounded.

4.2.3 Correspondences

We have seen how to build new spectral triples from old ones via the unbounded Kasparov product. However, it should be emphasized that in examples, the constructive Kasparov product often does not give the "correct" geometric representative of the underlying KK-class. This can be seen for instance in the example of a submersion, see Section 5.3. The missing terms are often related to curvature, and are of course "invisible" up to homotopy.

This eventually leads to a relative notion of KK-cycle between spectral triples, via (unbounded) *correspondences*. The notion of correspondences in noncommutative geometry were due to Connes and Skandalis [CS84], where they used geometric correspondences as cycles for (bounded) KK-theory.

Definition 4.19. Let $(\mathcal{A}, H_0, D_0)$ and $(\mathcal{B}, H_1, D_1)$ be spectral triples. A correspondence from $(\mathcal{B}, H_1, D_1)$ to $(\mathcal{A}, H_0, D_0)$ is a triple $(\mathcal{X}, S, D)$, where:

- $\mathcal{X}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule, together with a $\mathcal{B}$-valued inner product;
- $S\colon \mathcal{X} \to X$ is a symmetric operator such that $(\mathcal{A}, X, S) \in \Psi(A, B)$;
- $\nabla\colon \mathcal{X} \to X \otimes^{\mathrm{h}}_{\mathcal{B}} \Omega^1_{D_1}(\mathcal{B})$ is a connection;

(X is the completion of $\mathcal{X}$ under the inner product, A is the completion of $\mathcal{A}$, B is the completion of $\mathcal{B}$), such that:

- There is a unitary isomorphism of spectral triples $(\mathcal{A}, H_0, D_0) \simeq (\mathcal{A}, X \otimes_{\mathcal{B}} H_1, S \otimes 1 + 1 \otimes_\nabla D_1 + R)$.

In this definition, some extra analytic conditions have to be imposed. In particular, an abstract characterisation of the perturbation R is subject of ongoing research. The ultimate goal here is to obtain a flexible category of spectral triples in which the morphisms are correspondences.

5 Examples and outlook

5.1 Metric completeness

In differential geometry, there is a close connection between essential self-adjointness of first-order elliptic operators and metric completeness of the underlying Riemannian manifold. In addition, the Hopf-Rinow theorem states that for a Riemannian manifold, metric completeness and geodesic completeness are the same. In noncommutative geometry, metric completeness can be encoded in the existence of certain approximate identities. These in turn allow us to prove self-adjointness of symmetric operators. The following theorem is folkore, and formalises the notion of Friedrichs' mollifier on a manifold in the context of abstract operator theory. It came to the forefront in the study of the unbounded Kasparov product, first implicitly in [KL13] and more explicitly in [MR16]. A precise statement and proof can be found in [vdD18].

Theorem 5.1. *Let $D\colon X \supseteq \operatorname{Dom} D \to X$ be a symmetric operator. Suppose there is a sequence $(u_n) \subseteq \operatorname{End}^*_B(X)$, such that:*

- *For every n, u_n maps* $\operatorname{Dom} D^*$ *to* $\operatorname{Dom} D$ *and* $[D^*, u_n]$ *is bounded.*
- *For every $x \in X$, $u_n x \to x$ in norm.*
- *For every* $x \in \operatorname{Dom} D^*$, $[D^*, u_n]x \to 0$ *in norm.*

Then D is essentially self-adjoint and regular.

Proof. Let $x \in \operatorname{Dom} D^*$. Then

$$D^* x = \lim_n u_n D^* x = \lim_n D^* u_n x + [D^*, u_n]x = \lim_n D u_n x + [D^*, u_n]x.$$

So $\operatorname{Dom} D^* \subseteq \operatorname{Dom} D \subseteq \operatorname{Dom} D^*$, which proves self-adjointness. To obtain regularity, we invoke the local-global principle 3.10. For an irreducible representation $\pi : B \to \mathbb{B}(H_\pi)$, the operators $u_n \otimes 1 \in \mathbb{B}(X \otimes_B H_\pi)$ and $D \otimes 1$ satisfy the conditions of Theorem 5.1. Therefore $D \otimes 1$ is self-adjoint in each localisation, and hence it is regular as an operator in X. □

Definition 5.2. A *metric spectral triple* is a spectral triple $(\mathcal{A}, H, D)$ such that the Connes distance

$$\rho_D(\phi, \psi) := \sup \{ |\phi(a) - \psi(a)| \mid a \in \mathcal{A} \text{ self-adjoint}, \|[D, a]\| \leq 1 \},$$

is a metric and metrises the weak*-topology on the state space $\mathcal{S}(A)$ of $A := \overline{\mathcal{A}}$.

The idea of the Connes metric is related to the Monge–Kantorovich–Rubinstein–Wasserstein metrics that appear in probability theory. It needs emphasis that the Connes distance need not be a metric: this is part of the definition of metric spectral triple.

Definition 5.3 ([Hil10])**.** A *symmetric spectral triple* $(\mathcal{A}, H, D)$ over a C*-algebra A consists of:

- A $\mathbb{Z}/2$-graded Hilbert space $H = H_+ \oplus H_-$ such that A is represented by even bounded operators on H.
- D is a closed symmetric regular operator on H, such that $a(1 + D^* D)^{-1}$ is compact for all $a \in A$.
- A dense subalgebra $\mathcal{A} \subseteq A$ such that for all $a \in \mathcal{A}$: a maps $\operatorname{Dom} D^*$ to $\operatorname{Dom} D$ and $[D^*, a]$ extends to a bounded operator on H.

Theorem 5.4 ([MR16])**.** *Let $(\mathcal{A}, H, D)$ be a symmetric metric spectral triple. If there is an approximate unit for the C^*-algebra A such that*

$$u_n \colon \operatorname{Dom} D^* \to \operatorname{Dom} D \qquad \text{and} \qquad \sup_n \|[D, u_n]\| < \infty,$$

then $(\mathcal{S}(A), \rho_D)$ is complete.

Using the Hopf-Rinow theorem, one then readily deduces:

Theorem 5.5. *Let (M, g) be a smooth Riemannian manifold and $\mathrm{d} + \mathrm{d}^*$ its Hodge-deRham-Dirac operator. Suppose that there is an approximate unit $u_n \in \mathrm{C}_c^\infty(M)$ such that $[\mathrm{d} + \mathrm{d}^*, u_n] \to 0$ in norm. Then*

- $\mathrm{d} + \mathrm{d}^* : \mathrm{C}_c^\infty(M, \bigwedge T^* M) \to L^2(M, \bigwedge T^* M)$ *is essentially self-adjoint;*
- *M is geodesically complete.*

5.2 Operator spaces and operator modules

Operator spaces are closed subspaces of C^*-algebras, and their theory is surprisingly more subtle. A standard reference for operator space theory is the book [ER22], and for operator algebras and modules we refer to [BLM04].

5.2.1 Operator spaces and the Haagerup tensor product

Definition 5.6. An *operator space* $\mathcal{X}$ is a closed subspace of a C*-algebra A. There are canonical norms on $\mathbb{M}_n(\mathcal{X}) = \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{X}$ for each n via the inclusion into $\mathbb{M}_n(A)$.

A linear map $\phi\colon \mathcal{X} \to \mathcal{Y}$ between operator spaces is called *completely bounded*, resp. *completely contractive*, resp. *completely isometric*, if $1_n \otimes \phi\colon \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{X} \to \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{Y}$ are bounded, resp. contractive, resp. isometric.

The completely bounded norm (cb-norm) of a completely bounded map ϕ is defined as

$$\|\phi\|_{\mathrm{cb}} := \sup_n \|1_n \otimes \phi\colon \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{X} \to \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{Y}\|.$$

Example 5.7. Let E be a Hilbert C*-module over a C*-algebra B. Then E is an operator space, by identifying it with the $(2,1)$-corner of the *linking algebra* $\mathbb{K}_B(B \oplus E)$.

Remark 5.8. An operator space is a Banach space as it is a closed normed space. But it has more structures, namely, the norms on $\mathbb{M}_n(\mathcal{X})$ for each n. These are called the *matrix norms*. Two operator spaces may have the same underlying Banach space structure, whereas they are not isomorphism (i.e. completely isometrically isomorphic).

The Haagerup tensor product is the "correct tensor product" on operator spaces, possessing remarkable properties. It recovers the internal tensor products of Hilbert C*-modules.

Definition 5.9. Let $\mathcal{X}$ and $\mathcal{Y}$ be operator spaces. For each n, the *Haagerup norm* on $\mathbb{M}_n(\mathcal{X} \otimes^{\mathrm{alg}} \mathcal{Y})$ is defined by:

$$\|u\|_{\mathrm{h}} := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \;\middle|\; u = m\left(\sum_i x_i \otimes y_i\right),\, x_i \in \mathbb{M}_{n,p} \otimes \mathcal{X},\, y_i \in \mathbb{M}_{p,n} \otimes \mathcal{Y},\, p \in \mathbb{N} \right\},$$

where $m\colon \mathbb{M}_{n,p}(\mathbb{C}) \otimes \mathcal{X} \otimes \mathbb{M}_{p,n}(\mathbb{C}) \otimes \mathcal{Y} \to \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{X} \otimes \mathcal{Y}$ is $m(a \otimes x \otimes b \otimes y) := ab \otimes x \otimes y$, $\|x_i\|$ and $\|y_i\|$ are their operator norms as linear maps $\mathbb{C}^p \to \mathbb{C}^n$ and $\mathbb{C}^n \to \mathbb{C}^p$.

For concrete operator spaces $\mathcal{X} \subseteq \mathbb{B}(H)$ and $\mathcal{Y} \subseteq \mathbb{B}(K)$ the Haagerup norm on $\mathbb{M}_n(\mathcal{X} \otimes^{\mathrm{alg}} \mathcal{Y})$ can be equivalently described as

$$\left\| \sum_j x_j \otimes y_j \right\|_{\mathrm{h}} = \inf \left\{ \|v_i v_i^*\|^{1/2} \|w_i^* w_i\|^{1/2} \;\middle|\; \sum_i v_i \otimes w_i = \sum_j x_j \otimes y_j \right\}.$$

Definition 5.10. The *Haagerup tensor product* $\mathcal{X} \otimes^{\mathrm{h}} \mathcal{Y}$ of two operator spaces $\mathcal{X}$ and $\mathcal{Y}$ is the completion of the algebraic tensor product $\mathcal{X} \otimes^{\mathrm{alg}} \mathcal{Y}$ under the Haagerup norms for all n.

5.2.2 Operator algebras and operator modules

An *operator algebra* is a subalgebra of a C*-algebra. But it is better defined in terms of its intrinsic properties using the Haagerup tensor product.

Definition 5.11. An *operator algebra* is an operator space $\mathcal{A}$ which is an algebra, such that the multiplication induces a completely bounded map $\mathcal{A} \otimes^{\mathrm{h}} \mathcal{A} \to \mathcal{A}$.

Definition 5.12. An *operator module* over an operator algebra $\mathcal{A}$ is an operator space $\mathcal{M}$, which is a right module over $\mathcal{A}$ and such that the multiplication $\mathcal{M} \otimes^{\mathrm{h}} \mathcal{A} \to \mathcal{M}$ is completely bounded.

The research into the unbounded Kasparov product gave rise to the concept of an operator algebra with involution [Mes14, KL13] that was later developed in general [BKM18].

Definition 5.13. An *operator *-algebra* is an operator algebra $\mathcal{A}$ together with a completely isometric involution $\dagger\colon \mathcal{A} \to \mathcal{A}$. Note that even though an operator algebra $\mathcal{A}$ is a subalgebra of an ambient C*-algebra A, the involution $\dagger$ does not necessarily (in fact, never unless $\mathcal{A}$ is a C*-algebra) coincides with the involution * of A. This is emphasized by the different notation $\dagger$ instead of *.

Operator *-algebras and operator modules naturally arise from spectral triples (or more generally, unbounded Kasparov modules). Let $(\mathcal{A}, H, D)$ be a spectral triple, with grading γ. Recall that this implies that $\gamma \in \mathbb{B}(H)$ satisfies $\gamma^2 = 1$ and $\gamma = \gamma^*$, $\gamma D + D\gamma = 0$, a maps $\operatorname{Dom} D$ to $\operatorname{Dom} D$ and that $[D, a]$ is bounded for all $a \in \mathcal{A}$. Define

$$\pi_D \colon \mathcal{A} \to \mathbb{M}_2(\mathbb{B}(H)), \qquad a \mapsto \begin{pmatrix} a & 0 \\ [D, a] & \gamma a \gamma \end{pmatrix}.$$

Notice that

$$\pi_D(a)\pi_D(b) = \begin{pmatrix} a & 0 \\ [D, a] & \gamma a \gamma \end{pmatrix} \begin{pmatrix} b & 0 \\ [D, b] & \gamma b \gamma \end{pmatrix} = \begin{pmatrix} ab & 0 \\ [D, ab] & \gamma ab \gamma \end{pmatrix} = \pi_D(ab).$$

So π_D is an injective algebra homomorphism. We use π_D to identify $\mathcal{A}$ with the subalgebra $\pi_D(\mathcal{A})$ of $\mathbb{M}_2(\mathbb{B}(H))$. This equips $\mathcal{A}$ with the structure of an operator algebra. Note that as a Banach algebra, the norm on $\mathcal{A}$ is equivalent to the C^1-norm $\|a\|_1 = \|a\| + \|[D, a]\|$.

Note that $\pi_D(\mathcal{A})$ is not a *-subalgebra of $\mathbb{M}_2(\mathbb{B}(H))$ if we set * to be the involution of $\mathbb{M}_2(\mathbb{B}(H))$. But we have

$$\begin{aligned} \pi_D(a)^* = \begin{pmatrix} a & 0 \\ [D, a] & \gamma a \gamma \end{pmatrix}^* &= \begin{pmatrix} a^* & -[D, a^*] \\ 0 & \gamma a \gamma \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a^* & 0 \\ [D, a^*] & \gamma a^* \gamma \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pi_D(a^*) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

Therefore, if we endow $\pi_D(\mathcal{A})$ with the involution

$$\dagger \colon \pi_D(\mathcal{A}) \to \pi_D(\mathcal{A}), \qquad \pi_D(a) \mapsto \pi_D(a^*).$$

Then $(\mathcal{A}, \dagger) := (\pi_D(\mathcal{A}), \dagger)$ is an operator *-algebra, as $\dagger$ is completely isometric by (10). The relation in (10) is referred to as a standard form representation of $\mathcal{A}$, c.f. [BKM18, Definition 1.11].

The dense subspace $\operatorname{Dom} D$ is a Hilbert space under the inner product

$$\langle x, y \rangle_D := \langle x, y \rangle + \langle Dx, Dy \rangle,$$

and is therefore an operator space by Example 5.7. Notice that

$$\begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix} \begin{pmatrix} x \\ Dx \end{pmatrix} = \begin{pmatrix} ax \\ Dax \end{pmatrix}$$

holds for all $x \in \operatorname{Dom} D$. In particular, the map $\mathcal{A} \otimes^{\mathrm{h}} \operatorname{Dom} D \to \operatorname{Dom} D$, $a \otimes x \mapsto ax$ is completely bounded, where we identify $\mathcal{A}$ with the operator algebra $\pi_D(\mathcal{A})$ and equip $\operatorname{Dom} D$ with the operator space structure described above. This shows that $\operatorname{Dom} D$ is an operator module over $\mathcal{A}$.

5.3 Riemannian submersions

Riemannian submersions between spin$^{\mathrm{c}}$-manifolds can be described nicely within the framework of unbounded KK-theory. This was written down explicitly in [KvS18].

Let M and B be smooth, compact Riemannian manifolds without boundary. Let $\pi \colon M \to B$ be a smooth surjective map. Denote by $\mathfrak{X}(M)$, resp. $\mathfrak{X}(B)$, the space of smooth vector fields on M, resp. on B. They are $\mathrm{C}^\infty(M)$, resp. $\mathrm{C}^\infty(B)$-modules, and moreover, endowed with the $\mathrm{C}^\infty(M)$-valued, resp. $\mathrm{C}^\infty(B)$-valued Hermitian inner products that come from the Riemannian metrics on the corresponding manifolds.

The map $\pi \colon M \to B$ induces a left $\mathrm{C}^\infty(B)$-module structure on $\mathrm{C}^\infty(M)$ via pullback.

Definition 5.14. We say π is a *Riemannian submersion*, if the $\mathrm{C}^\infty(M)$-module map

$$\mathrm{d}\pi\colon \mathfrak{X}(M) \to \mathfrak{X}(B) \otimes_{\mathrm{C}^\infty(B)} \mathrm{C}^\infty(M), \qquad \mathrm{d}\pi(X)(f) := X(f \circ \pi)$$

is surjective, and $\mathrm{d}\pi|_{(\ker \mathrm{d}\pi)^\perp}$ is an isometric isomorphism. Here

$$(\ker \mathrm{d}\pi)^\perp := \{X \in \mathfrak{X}(M) \mid \langle X, Y\rangle = 0 \text{ for all } Y \in \ker \mathrm{d}\pi\}.$$

The Riemannian structure on M yields a decomposition of $\mathfrak{X}(M)$ into vertical part (which is canonical and does not depend on the metric) and horizontal part (depending on the metric).

Now assume furthermore than M and B are spin$^{\mathrm{c}}$-manifolds. Then there are Dirac spectral triples

$$(\mathrm{C}^\infty(M), L^2(M, \$_M), D_M), \quad (\mathrm{C}^\infty(B), L^2(B, \$_B), D_B)$$

for M and B. Here $L^2(M, \$_M)$ and $L^2(B, \$_B)$ are the L^2-completion of the sections of the corresponding spinor bundles, and D_E and D_B are their spin Dirac operators.

Under the assumption that π is a Riemannian submersion, the $\mathrm{C}^\infty(M)$-module of smooth sections $\Gamma^\infty(M, \$_M)$ can be decomposed as the (tensor product) of two submodules: a "horizontal" part coming from the pullback of $\Gamma^\infty(B, \$_B)$ along π; and a "vertical" part $\mathcal{X}$.

The vertical submodule $\mathcal{X}$ is a $(\mathrm{C}^\infty(M), \mathrm{C}^\infty(B))$-bimodule and can be completed into a $(\mathrm{C}(M), \mathrm{C}(B))$-bimodule X. Moreover, $\mathcal{X}$ possesses a natural "vertical Dirac operatos" D_{v}, and a natural "vertical connection"

$$\nabla^{\mathcal{X}}\colon \mathcal{X} \to X \otimes^{\mathrm{h}}_{\mathrm{C}^\infty(M)} \Omega^1_{D_M}(\mathrm{C}^\infty(M)).$$

This vertical module together with the vertical Dirac operator defines an unbounded Kasparov module, factorising the Dirac spectral triple on M as the unbounded Kasparov product of the Dirac spectral triple on B, and the vertical Kasparov module. More precisely:

Theorem 5.15 ([KvS18, Theorem 24])**.** $(\mathrm{C}^\infty(M), \mathcal{X}, D_{\mathrm{v}}) \in \Psi(\mathrm{C}(M), \mathrm{C}(B))$. *There is a unitary isomorphism*

$$(\mathrm{C}^\infty(M), L^2(M, \$_M), D_M) \simeq (\mathcal{X} \otimes^{\mathrm{h}}_{\mathrm{C}^\infty(B)} L^2(B, \$_B), D_{\mathrm{v}} \otimes 1 + 1 \otimes_{\nabla^{\mathcal{X}}} D_B + \mathrm{c}(\Omega))$$

of spectral triples, where $\mathrm{c}(\Omega)$ *is the Clifford representation of the curvature of* $\pi\colon M \to B$.

As a corollary, the class $[\mathrm{C}^\infty(M), L^2(M, \$_M), D_M] \in \mathrm{KK}(\mathrm{C}(M), \mathbb{C})$ *is the Kasparov product of*

$$[\mathrm{C}^\infty(M), \mathcal{X}, D_{\mathrm{v}}] \in \mathrm{KK}(\mathrm{C}(M), \mathrm{C}(B)) \quad \textit{and} \quad [\mathrm{C}^\infty(B), L^2(B, \$_B), D_B] \in \mathrm{KK}(\mathrm{C}(B), \mathbb{C}).$$

5.4 Non-isometric actions on Riemannian manifolds

In this final section, we will describe some (counter)examples, where the naturally assigned geometric data cannot be made into a correspondence. This example is a blueprint for many cases where problems arise with the commutators with $1 \otimes_\nabla T$.

As discussed in Section 4.2.2 and Section 4.2.3: in order to construct a new spectral triple on A from a spectral triple $(\mathcal{B}, H, T)$, we need to find a connection ∇ and build a correspondence $(\mathcal{X}, S, \nabla)$, and expect that

$$(\mathcal{A}, X \otimes_B H, S \otimes 1 + 1 \otimes_\nabla T)$$

is a spectral triple. This requires that $[1 \otimes_\nabla T, a]$ is bounded for all $a \in \mathcal{A}$. Unfortunately, it fails for non-isometric actions on Riemannian manifolds.

Let (M, g) be a compact Riemannian manifold, and $\alpha\colon M \to M$ be a diffeomorphism. Then α generates an action of $\mathbb{Z}$ on $\mathrm{C}(M)$ and on $\mathrm{C}^\infty(M)$:

$$\alpha(n)f(x) := f(\alpha^{-n}(x)).$$

This also gives a crossed product C*-algebra $\mathrm{C}(M) \rtimes_\alpha \mathbb{Z}$ and a dense subalgebra $\mathrm{C}^\infty(M) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}$ of it. In the following, we will construct an unbounded Kasparov module which represents the Pimsner–Voiculescu boundary map in KK-theory.

Set $\mathcal{X} := \mathrm{C_c}(\mathbb{Z}) \otimes^{\mathrm{alg}} \mathrm{C}(M)$. Then $\mathcal{X}$ is a $(\mathrm{C}^\infty(M), \mathrm{C}^\infty(M))$-bimodule, and its completion $X := \ell^2(\mathbb{Z}) \otimes \mathrm{C}(M)$ is a $(\mathrm{C}(M), \mathrm{C}(M))$-bimodule: the right module structures of $\mathcal{X}$ and X are given by multiplication, and the left module structures are implemented by the *-homomorphism

$$\pi\colon \mathrm{C}(M) \to \mathrm{End}^*_{\mathrm{C}(M)}(X), \qquad \pi(f)(e_n \otimes \psi) = e_n \otimes \pi(\alpha(n)f)\psi.$$

With a slight abuse of notation, we just write

$$\alpha\colon \mathbb{Z} \to \mathrm{End}^*_{\mathrm{C}(M)}(X), \qquad \alpha(k)(e_n \otimes \psi) = e_{n+k} \otimes \psi.$$

Then (π, α) is a covariant representation of the C*-dynamical system $(\mathrm{C}(M), \mathbb{Z}, \alpha)$. Therefore, it generates a *-homomorphism $\mathrm{C}(M) \rtimes_\alpha \mathbb{Z} \to \mathrm{End}^*_{\mathrm{C}(M)}(X)$ and equips X with a $(\mathrm{C}(M) \rtimes_\alpha \mathbb{Z}, \mathrm{C}(M))$-bimodule structure.

Assigned to every C*-algebra of the form $A \rtimes_\alpha \mathbb{Z}$, there is a Pimsner–Voiculescu sequence in K-theory

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{1-\alpha_*} & \mathrm{K}_0(A) & \xrightarrow{\iota_*} & \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \\ \partial \uparrow & & & & \downarrow \partial \\ \mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[\iota_*]{} & \mathrm{K}_1(A) & \xleftarrow[1-\alpha_*]{} & \mathrm{K}_1(A). \end{array}$$

The boundary maps $\partial\colon \mathrm{K}_*(A \rtimes_\alpha \mathbb{Z}) \to \mathrm{K}_{*+1}(A)$ are implemented by a class in the odd KK-theory $\mathrm{KK}_1(A \rtimes_\alpha \mathbb{Z}, A)$.

Define the unbounded operator S on X with domain $\mathcal{X}$:

$$S\colon \mathrm{C_c}(\mathbb{Z}) \otimes \mathrm{C}(M) \to \ell^2(\mathbb{Z}) \otimes \mathrm{C}(M), \qquad S(e_n \otimes f) = n e_n \otimes f.$$

Proposition 5.16. $(\mathrm{C}^\infty(M) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}, X, S)$ *is an unbounded odd Kasparov* $(A \rtimes_\alpha \mathbb{Z}, A)$*-module. It represents the Pimsner–Voiculescu boundary map* ∂*.*

Suppose that a spectral triple $(\mathrm{C}^\infty(M), L^2(M), D)$ on M is given. Can we turn $(\mathrm{C}^\infty(M) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}, X, S)$ into a self-correspondence for $(\mathrm{C}^\infty(M), L^2(M), D)$? This means that we need to find a connection

$$\nabla\colon \mathcal{X} \to \ell^2(\mathbb{Z}) \otimes \Omega^1_D(\mathrm{C}^\infty(M)).$$

We have a natural choice:

Proposition 5.17. *Set* $\nabla(e_n \otimes f) := e_n \otimes [D, f]$*. Then:*

- $1 \otimes_\nabla D$ *is essentially self-adjoint.*
- *The operators*

$$\begin{pmatrix} 0 & S \otimes 1 \\ S \otimes 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\mathrm{i} \otimes_\nabla D \\ \mathrm{i} \otimes_\nabla D & 0 \end{pmatrix}$$

  anticommute.

Therefore, the operator

$$\begin{pmatrix} 0 & S \otimes 1 - \mathrm{i} \otimes_\nabla D \\ S \otimes 1 + \mathrm{i} \otimes_\nabla D & 0 \end{pmatrix}$$

is self-adjoint and has compact resolvent.

However, we find that:

$$[1 \otimes_\nabla D, \pi(f)](e_n \otimes \psi) = e_n \otimes [D, \alpha(n)f]\psi.$$

Therefore,

$$\|[1 \otimes_\nabla D, \pi(f)]\| = \sup_n \|[D, \alpha(n)f]\|.$$

So the commutator blows up as long as $\|\mathrm{d}\alpha\| \neq 1$ somewhere. Namely, unless α is isometric, we are unable to turn $(\mathrm{C}^\infty(M) \rtimes_\alpha^{\mathrm{alg}} \mathbb{Z}, X, S)$ into a correspondence using the naturally assigned operator ∇.

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