C*-algebras of Bratteli diagrams and foliations of translation surfaces

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Contents

Introduction

The following document is an (unfinished) note for the second part of a reading seminar on foliations[1], which was held at Leiden university in 2023 fall. The first part of the seminar covered some basic definitions: foliations, their holonomy groupoids and C*-algebras. The second part of the seminar was devoted to a careful reading of the preprint [PT22] by Putnam and Treviño.

Lindsey and Treviño [LT16] have constructed translation surfaces from bi-infinite Bratteli diagrams using combinatorial methods. These were further studied by Putnam and Treviño with operator algebraic techniques in [PT22], in which they display explicit relations between the groupoid C*-algebra of the Bratteli diagram, and the C*-algebra constructed from suitable foliations on the translation surface. This allows them to compute the K-theory of the aforementioned C*-algebras.

I am indebted to Olga Lukina, Dimitris Gerotogiannis, Malte Leimbach and Yufan Ge for giving these interesting talks.

List of symbols

The article [PT22] of Putnam and Treviño is quite involved. Moreover it consists of a substantial number of symbols, making it even more difficult to read. For our convenience, a collection of symbols used in this paper is provided below. I have been trying to make it complete, but some symbols which are used only locally might be dropped out.

Bratteli diagrams

Let $\mathcal{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram with order $\leq_s$ and $\leq_r$ (1.12). Let $v \in V$ be a vertex and $x \in X_\mathcal{B}$ be a (bi-infinite) path of $\mathcal{B}$.

Symbol	Meaning	Reference
$X_\mathcal{B}$	Space of bi-infinite paths in $\mathcal{B}$.	1.10
$X_\mathcal{B}^\pm$	Space of left/right infinite paths in $\mathcal{B}$.	3.6
$X_v^\pm$	Space of left/right infinite paths in $\mathcal{B}$ starting/ending at v.	2.5
$X_\mathcal{B}^{s\text{-max}}$, $X_\mathcal{B}^{r\text{-max}}$, $X_\mathcal{B}^{s\text{-min}}$, $X_\mathcal{B}^{r\text{-min}}$	Space of s-maximal/r-maximal/ s-minimal/r-minimal paths in $\mathcal{B}$.	2.14
$X_\mathcal{B}^{\text{ext}}$	Extreme points of $X_\mathcal{B}$. $X_\mathcal{B}^{\text{ext}} := X_\mathcal{B}^{s\text{-max}} \cup X_\mathcal{B}^{r\text{-max}} \cup X_\mathcal{B}^{s\text{-min}} \cup X_\mathcal{B}^{r\text{-min}}$.	2.14
$\partial_s X_\mathcal{B}, \partial_r X_\mathcal{B}$	s-boundary of $X_\mathcal{B}$/r-boundary of $X_\mathcal{B}$. $\partial_s X_\mathcal{B} := \{x \in X_\mathcal{B} \mid x$ has an s-successor or an s-predecessor$\}$. $\partial_r X_\mathcal{B} := \{x \in X_\mathcal{B} \mid x$ has an r-successor or an r-predecessor$\}$.	2.15
$\partial X_\mathcal{B}$	Boundary of $X_\mathcal{B}$. $\partial X_\mathcal{B} := \partial_s X_\mathcal{B} \cap \partial_r X_\mathcal{B}$.	2.18
Δ_s, Δ_r	$\Delta_s \colon \partial_s X_\mathcal{B} \to \partial_s X_\mathcal{B}$ sending x to its s-successor or s-predecessor. $\Delta_r \colon \partial_r X_\mathcal{B} \to \partial_r X_\mathcal{B}$sending x to its r-successor or r-predecessor.	2.17
$\Sigma_\mathcal{B}$	Singular points of $X_\mathcal{B}$. $\Sigma_\mathcal{B} := \{x \in \partial X_\mathcal{B} \mid \Delta_r \circ \Delta_s(x) \neq \Delta_s \circ \Delta_r(x)\}$.	2.18

[1] https://ncg-leiden.github.io/foliation2023. The complete note can be found here.

$Y_{\mathcal{B}}$	$Y_{\mathcal{B}} := X_{\mathcal{B}} \setminus (X_{\mathcal{B}}^{\text{ext}} \cup \Sigma_{\mathcal{B}})$.	2.20
$S_{\mathcal{B}}^s, S_{\mathcal{B}}^r$	$S_{\mathcal{B}}^s := Y_{\mathcal{B}} \;/\; x \sim \Delta_s(x),\ x \in Y_{\mathcal{B}} \cap \partial_s X_{\mathcal{B}}$. $S_{\mathcal{B}}^r := Y_{\mathcal{B}} \;/\; x \sim \Delta_r(x),\ x \in Y_{\mathcal{B}} \cap \partial_r X_{\mathcal{B}}$.	3.1
$S_{\mathcal{B}}$	$S_{\mathcal{B}} := Y_{\mathcal{B}} \Big/ \begin{array}{l} x \sim \Delta_s(x) \text{ if } x \in \partial_s X_{\mathcal{B}} \cap Y_{\mathcal{B}}; \\ x \sim \Delta_r(x) \text{ if } x \in \partial_r X_{\mathcal{B}} \cap Y_{\mathcal{B}}. \end{array}$	2.20
$T_N^+(x), T_N^-(x)$	$T_N^+(x) := \{y \in X_{\mathcal{B}} \mid y_n = x_n \text{ for all } n > N\}$. $T_N^-(x) := \{y \in X_{\mathcal{B}} \mid y_n = x_n \text{ for all } n \leq N\}$.	2.9
$T^+(x), T^-(x)$	Tail equivalence classes. $T^+(x) := \bigcup_{N\in\mathbb{Z}} T_N^+(x)$. $T^-(x) := \bigcup_{N\in\mathbb{Z}} T_N^-(x)$.	2.9
$x_{(n,+\infty)}, x_{[m,n)}$	$x_{(n,+\infty)} := (\ldots, x_{n+2}, x_{n+1})$. $x_{[m,n)} := (x_{n-1}, x_{n-2}, \ldots, x_m)$.	2.10
$E_{m,n}$	$E_{m,n} := \prod_{m<i\leq n} E_i = \{\text{Finite paths from } V_m \text{ to } V_n\}$.	2.10
$E_{m,n}^Y$	Finite paths $p \in E_{m,n}$ with: - p is neither s-maximal, s-minimal, r-maximal nor r-minimal. - $X_{s(p)}^- p X_{r(p)}^+ \subseteq Y_{\mathcal{B}}$.	3.2
$E_{m,n}^s$	Set of pairs $p = (p_1, p_2)$ satisfying: - $p_i \in E_{m,n}^Y$. - p_2 is an s-successor of p_1.	3.25
$E_{m,n}^{s/r}$	Set of quadruples $p = (p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2})$ satisfying: - $p_{i,j} \in E_{m,n}^Y$. - $p_{i,j+1}$ is an s-successor of $p_{i,j}$. - $p_{i+1,j}$ is an r-successor of $p_{i,j}$.	3.3

Inductive systems

Inductive system	Inductive limit	Reference
$A_{m,n}^+$	$A_{\mathcal{B}}^+$	3.17
$A_{m,n}^{Y+}$	$A_{\mathcal{B}}^{Y+}$	3.19
$AC_{m,n}^+$		3.20
$AC_{m,n}^{Y+}$		3.20
$B_{-n,n}^+$	$B_{\mathcal{B}}^+$	3.23
$C_{-n,n}^+$	$C_{\mathcal{B}}^+$	3.30
$G_{m,n}$		3.25
$H_{m,n}$		3.32

Groupoids

The groupoids in [PT22, §7] as well as their relations are listed below. By $\mathcal{G} \hookrightarrow \mathcal{H}$ I mean $\mathcal{G}$ is an open subgroupoid of $\mathcal{H}$.

$$
\begin{tikzcd}
 & T^+(X_{\mathcal{B}}) & & \\
 & T^+(Y_{\mathcal{B}}) \arrow[u, hook, "1"] \arrow[dl, "\pi^r \times \pi^r"'] & T^\sharp(Y_{\mathcal{B}}) \arrow[l, hook', "2"'] \arrow[dr, "\pi^s \times \pi^s"] & \\
T^+(S^r_{\mathcal{B}}) & & & T^\sharp(S^s_{\mathcal{B}}) \arrow[dl, "\rho^r \times \rho^r"] \\
 & & T^\sharp(S_{\mathcal{B}}) & \\
 & & \mathcal{F}^+_{\mathcal{B}} \arrow[u, hook, "3"'] &
\end{tikzcd}
$$

(1) $T^+(Y_{\mathcal{B}}) := T^+(X_{\mathcal{B}})\big|_{Y_{\mathcal{B}}}^{Y_{\mathcal{B}}}$.

(2) $T^\sharp(Y_{\mathcal{B}}) := \big\{(x, y) \in T^+(Y_{\mathcal{B}}) \,\big|\, \text{If } x, y \in \partial_s X_{\mathcal{B}}, \text{ then } (\Delta_s(x), \Delta_s(y)) \in T^+(Y_{\mathcal{B}})\big\}$.

(3) $\mathcal{F}^+_{\mathcal{B}} := \Big\{(x, y) \in T^\sharp(S_{\mathcal{B}}) \,\Big|\, x, y \text{ are in the same connected component}\Big\}$.

C*-algebras

The groupoids in the diagram above yield their groupoid C*-algebras in the diagram below. Besides that, several inductive system are introduced in [PT22, §8], displaying these groupoid C*-algebras as AF-algebras and providing useful short exact sequences to compute their K-theory.

$$
\begin{tikzcd}
 & A^+_{\mathcal{B}} := \mathrm{C}^*(T^+(X_{\mathcal{B}})) & & \\
 & A^{Y+}_{\mathcal{B}} := \mathrm{C}^*(T^+(Y_{\mathcal{B}})) \arrow[u, hook, "\text{hereditary}"] \arrow[dl, "\sim"] & \mathrm{C}^*(T^\sharp(Y_{\mathcal{B}})) \arrow[l, hook'] & \\
\mathrm{C}^*(T^+(S^r_{\mathcal{B}})) & & & B^+_{\mathcal{B}} := \mathrm{C}^*(T^\sharp(S^s_{\mathcal{B}})) \arrow[ul, hook'] \arrow[dl, "\sim"] \\
 & & \mathrm{C}^*(T^\sharp(S_{\mathcal{B}})) & \\
 & & C^+_{\mathcal{B}} := \mathrm{C}^*(\mathcal{F}^+_{\mathcal{B}}) \arrow[u, hook] &
\end{tikzcd}
$$

1 Translation surfaces and bi-infinite Bratteli diagrams

Translation surfaces are surfaces obtained by identifying several edges of polygons in the Euclidean plane. They are useful as models of dynamical systems on the unit interval $[0, 1]$. The *Bratteli diagrams* are first used by operator algebraists for studying AF-algebras, but later endued with dynamical meanings after equipped with an order (on the path space).

An ordered Bratteli diagram therefore models a Cantor dynamical system. There is a well-known strategy to pass from a dynamical system on $[0, 1]$ to one on the Cantor set by adding (countably many) limit points and equip the latter with an invariant measure. But for long it is unclear how to reverse this process. The recent paper [PT22] of Putnam and Treviño provides a solution.

In this first lecture, we provide their definitions and work out some practical examples, covering those from [PT22, §2–4].

1.1 Translation surfaces

A translation surface can be defined in various different ways. We start with a constructive definition.

Definition 1.1 (First definition, constructive)**.** Let $\mathcal{P}$ be an at most countable family of polygons in $\mathbb{R}^2$. For each polygon $P \in \mathcal{P}$, let $\mathcal{E}(P)$ denote the set of all line segments of the boundary of P. For each $e \in \mathcal{E}(P)$, denote the inward normal unit vector by n_e. Assume that:

There is a pairing (i.e. an involutive map)

$$f\colon \bigcup_{P\in\mathcal{P}} \mathcal{E}(P) \to \bigcup_{P\in\mathcal{P}} \mathcal{E}(P)$$

such that:

- $f(e)$ differs from e by some translation τ_e for every $e \in \mathcal{E}(P)$.
- $n_{f(e)} = -n_e$.

Then a *translation surface* is defined as the quotient space

$$M := \left. \coprod_{P\in\mathcal{P}} P \middle/ e \sim f(e) \right.$$

and with all vertices of degree greater than 2 removed.

Example 1.2. Figure 1.1 gives an example of a translation surface, obtained by identifying those edges of the same color of three rectangles. The resulting surface M is not compact as it has a single puncture, which corresponds to an end of the surface. Removing this point gives a compact surface of finite genus.

We may compute its genus using Euler characteristic

$$\chi = v - e + f = 2 - 2g$$

where v (e, f) is the number of vertices (edges, faces) and g is the genus of the surface. There is a unique vertice: notice that in the figure, the point A is identified with C' after gluing together the red edges, and then with C (through the blue edges) $\longrightarrow A'$ (green) $\longrightarrow D'$ (yellow) $\longrightarrow B$ (red) $\longrightarrow B'$ (blue) $\longrightarrow D$ (green).

Clearly there are four edges (taking also the dashed edges into account) and three faces. So

$$\chi = 1 - 6 + 3 = 2 - 2g \quad\Longrightarrow\quad g = 2.$$

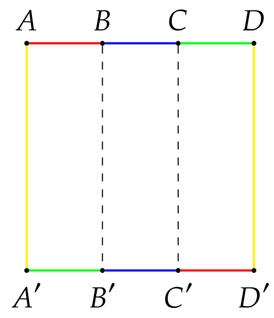


Figure 1.1: An example of a translation surface

The point we have removed is called a *cone-angle singularity*, as it does not have an Euclidean neighbourhood: wrapping around this vertice gives an angle of 6π. A formal definition is that there exists $k \geq 2$ such that the angle around this point is $2\pi k$.

Definition 1.3. A translation surface S has *finite type* if S has finite area, and is isomorphic to a compact Riemann surface after a finite number of points removed. S is said to of *infinite type* if it does not have finite type.

Now we provide an "intrinsic" definition of translation surfaces.

Definition 1.4 (Second definition, by translation atlas)**.** A *translation atlas* on a topological surface S is an atlas $\{(U_i, \varphi_i \colon U_i \to \mathbb{C})\}$ of S such that the transition maps $\varphi_i \circ \varphi_j^{-1}$ are translations on their domains.

A *translation surface* is a topological surface together with a translation atlas thereon.

Example 1.5 (Infinite staircases)*.* Figure 1.2a is known as the *infinite staircases*, which is obtained by identifying the edges that are facing each other. There are four cone-angle singularies A, B, C, D, each of which having infinite degree. It is not hard to see that the surface has infinite genus.

Example 1.6 (Chamanara surface)*.* Figure 1.2b displays the baker's (or Chamanara) surface introduced in [Cha04]. Consider a square with sides of length 1. Divide each side consecutively into segments, with the n-th segment of length $\frac{1}{2^n}$. Those segments are viewed as edges of a polygon, and those of the same length which lie on opposite sides are identified. This gives a translation surface, which has finite area because it comes from a square of area 1; and infinite genus. It has only one singularity, but a quite singular one, not even an infinite cone-angle singularity because there is no (infinite) covering with Euclidean disks. Such a singularity is called *wild*.

1.2 Translation flows

Let $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. For each $z \in \mathbb{C}$ we may define a *parallel flow* $(F^t_{\mathbb{C},\vartheta})_{t\in[0,\infty)}$ at z:

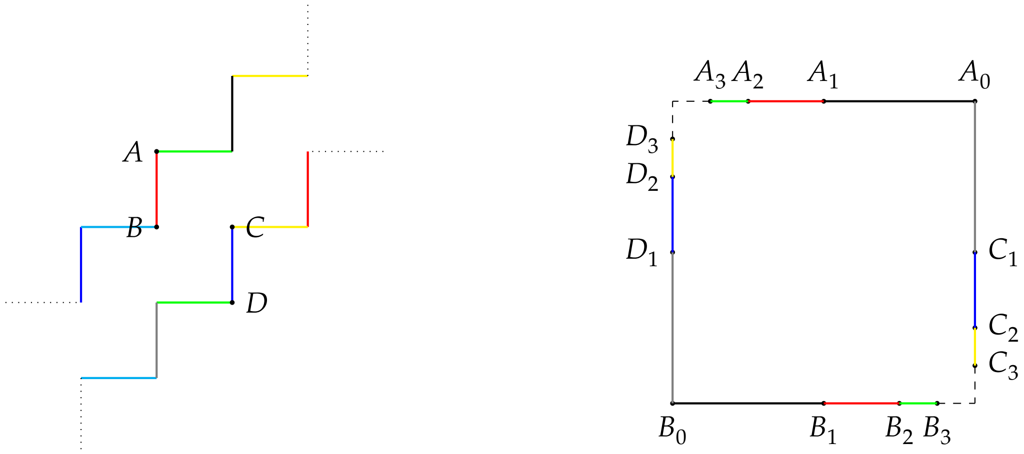
$$F^t_{\mathbb{C},\vartheta} \colon \mathbb{C} \to \mathbb{C} \qquad z \mapsto z + t\exp(i\vartheta).$$

The flows generate the vector field

$$X_{\mathbb{C},\vartheta} = \left.\frac{\partial F^t_{\mathbb{C},\vartheta}}{\partial t}\right|_{t=0}(z).$$

Let S be a translation surface with a translation atlas $\{(U_i, \varphi_i \colon U_i \to \mathbb{C})\}$. Pulling back the vector field $X_{\mathbb{C},\vartheta}$ along charts gives vector fields $X_{S,\vartheta}$ on each chart domain. These vector fields can be glued together because the charts differ only by translations. Thus $X_{S,\vartheta}$ is a vector field defined on the whole of S.

Definition 1.7. A *translation flow* is the collection of maximal integral curves of $X_{S,\vartheta}$.



(a) The infinite staircases

(b) The baker's (or Chamanara) surface

Figure 1.2: Examples of translation surfaces

1.3 Bratteli diagrams

1.3.1 Motivation

The identification of segments on opposite sides of the Chamanara surface (Example 1.6) is also called the *Kakutani–von Neumann map*, which gives an infinite interval exchange transformation (see [Put92]) on $[0,1)$. Namely, the dynamics is generated by the translations

$$[\tfrac{1}{2},1) \to [0,\tfrac{1}{2}), \quad [\tfrac{1}{4},\tfrac{1}{2}) \to [\tfrac{1}{2},\tfrac{3}{4}), \quad [\tfrac{1}{8},\tfrac{1}{4}) \to [\tfrac{3}{4},\tfrac{7}{8}), \quad \ldots$$

Adding the limit points yields a dynamical system on the Cantor set. The new dynamical system is not conjugate to the interval exchange transformation, but since the limit points are countable, we are still able to equip this Cantor dynamical system with an invariant measure. The Kakutani–von Neumann map is a simple example of an infinite interval exchange, and more complicated cases might be found in [BL23] and the references therein.

A Cantor dynamical system gives an ordered Bratteli diagram using the *Kakutani–Rokhlin partitions* (see [Bru22, Chapter 5]). The resulting ordered Bratteli diagrams are not unique, yet equivalent in a suitable sense. Conversely, starting from an ordered Bratteli diagram one can construct a Cantor dynamical system which is inverse to the previous construction. This is a well-known result from [HPS92]. That says, we have a beautiful dictionary

$$\text{“Dynamics on the Cantor set”} \quad \overset{\sim}{\longleftrightarrow} \quad \text{“Dynamics of an ordered Bratteli diagram”.}$$

The simplest case for the Cantor dynamical system generated by the Kakutani–von Neumann map on the unit interval is given by the *adding machine* (Figure 1.3). The dynamics on a Cantor set can be described by its Bratteli diagram, and the transformation is given by the *Vershik map*, which roughly speaking sends every infinite path to its successor, and the maximal path to the minimal path.

1.3.2 A glimpse of ordered bi-infinite Bratteli diagrams

A Bratteli diagram, roughly speaking, is a directed graph[2] whose set of vertices is $\mathbb{N}$-graded (as a set) by finite subsets, and an edge increases the degree by 1. A bi-infinite Bratteli diagram is similar

[2]For more on graph theory, see Yufan's talk on graph C*-algebras in [NCG22b], or Adam's talk on topological graphs in [NCG22a].

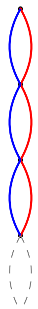


Figure 1.3: The adding machine, which is an ordered Bratteli diagram

but its set of vertices is $\mathbb{Z}$-graded. More precisely, we have:

Definition 1.8 ([PT22, Definition 2.1])**.** A *Bratteli diagram* is a quadruple $\mathcal{B} = (V, E, r, s)$, where V and E are two sets and $r, s \colon E \rightrightarrows V$ are *surjective* maps between them, such that V and E are disjoint unions of non-empty *finite* sets:

- $V = \coprod_{n \in \mathbb{N}_{\geq 0}} V_n$, and $V_0 = \{v_0\}$ is a singleton.
- $E = \coprod_{n \in \mathbb{N}_{\geq 1}} E_n$, such that $r(E_n) = V_n$ and $s(E_n) = V_{n-1}$.

An element in V is called a *vertice* and an element in E is called an *edge*. The maps r and s are called the *range* and *source* maps.

Definition 1.9 ([PT22, Definition 2.2])**.** A *bi-infinite Bratteli diagram* is a quadruple $\mathcal{B} = (V, E, r, s)$, defined similarly as in Definition 1.8, but replacing the conditions for V and E by:

- $V = \coprod_{n \in \mathbb{Z}} V_n$.
- $E = \coprod_{n \in \mathbb{Z}} E_n$, such that $r(E_n) = V_n$ and $s(E_n) = V_{n-1}$.

Figure 1.4 is an example of a Bratteli diagram, which is a *stationary* one as it has repeated pattern.

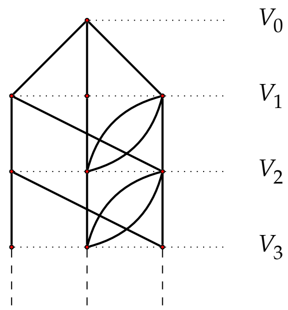


Figure 1.4: An example of a Bratteli diagram

Definition 1.10 ([PT22, Definition 3.1])**.** Let $\mathcal{B}$ be a Bratteli diagram or a bi-infinite Bratteli diagram. Denote by $X_{\mathcal{B}}$ the set of *infinite paths* in $\mathcal{B}$. That is, an infinite word

$$x = (\dots, x_{i+1}, x_i, \dots, x_2, x_1) \quad \in \quad \prod_{i \in \mathbb{N}_{\geq 1}} E_i,$$

or a bi-infinite word

$$x = (\ldots, x_{i+1}, x_i, \ldots, x_2, x_1, \ldots) \quad \in \quad \prod_{i \in \mathbb{Z}} E_i,$$

such that $s(x_{i+1}) = r(x_i)$ for all i.

We equip $X_{\mathcal{B}}$ with the Tychonoff topology on $\prod_{i \in \mathbb{N}_{\geq 1}} E_i$ or $\prod_{i \in \mathbb{Z}} E_i$.

Let $\mathcal{B}$ be a Bratteli diagram or a bi-infininite Bratteli diagram. We are going to define a partial order on $X_{\mathcal{B}}$. This requires an order, such that all edges outgoing a fixed vertice are comparable. This induces a partial order on $X_{\mathcal{B}}$, in which two paths are comparable if they are *tail equivalent*.

Definition 1.11 ([PT22, Definition 3.5])**.** Let $\mathcal{B}$ be a Bratteli diagram. Let $x = (x_i)_{i \in \mathbb{N}_{\geq 1}}$ and $y = (y_i)_{i \in \mathbb{N}_{\geq 1}}$ be infinite paths of $\mathcal{B}$. We say they are *tail equivalent*, if there exists some $n \in \mathbb{N}_{\geq 1}$, such that $e_i = f_i$ for all $i \geq n$.

Similar, let $\mathcal{B}$ be a bi-infinite Bratteli diagram. Let $x = (x_i)_{i \in \mathbb{Z}}$ and $y = (y_i)_{i \in \mathbb{Z}}$ be infinite paths of $\mathcal{B}$. We say they are *left* (or *right*) *tail equivalent*, if there exists some $n \in \mathbb{Z}$, such that $e_i = f_i$ for all $i \geq n$ (or $i \leq n$).

Below we will define an order on the path space of a bi-infinite Bratteli diagram. The definition can be easily translated to Bratteli diagrams as well.

Definition 1.12 ([PT22, Definition 2.10])**.** An *ordered* bi-infinite Bratteli diagram is a bi-infinite Bratteli diagram $\mathcal{B} = (V, E, r, s)$ together with partial orders $\leq_s$, $\leq_r$ on E, such that for every pair of edges $e, e' \in E$:

- e and e' are $\leq_s$ comparable, if $s(e) = s(e')$.
- e and e' are $\leq_r$ comparable, if $r(e) = r(e')$.

That means that $\leq_s$ (resp. $\leq_r$) is a partial order which restricts to a linear order on $s^{-1}(v)$ (resp. $r^{-1}(v)$) for every $v \in V$.

We write $e <_s e'$ (resp. $e <_r e'$) if $e \leq_s e'$ (resp. $e \leq_r e'$) and $e \neq e'$.

Definition 1.13 ([PT22, Lemma 4.4])**.** Let $\mathcal{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram with partial orders $\leq_s$ and $\leq_r$. We define partial orders $\leq_s$ and $\leq_r$ on $X_{\mathcal{B}}$ as follows. Given infinite paths $x = (x_i)_i$ and $y = (y_i)_i$, we say:

- $x \leq_r y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \geq n$ and $x_n \leq_r y_n$.
- $x \leq_s y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \leq n$ and $x_n \leq_s y_n$.

Notice that this means that x and y are comparable in $\leq_r$ (resp. $\leq_s$) iff they are left (resp. right) tail equivalent.

2 The surface associated with a bi-infinite Bratteli diagram

In this lecture, we will work towards the construction of a translation surface from an ordered, bi-infinite Bratteli diagram $\mathcal{B}$. More details and properties will be provided in the next talk of Malte. Along the way, we will also construct measures on the "leaves" of the path space $X_{\mathcal{B}}$, which are later utilised to build a Haar system on a topological groupoid coming from $\mathcal{B}$.

The idea of [PT22] is very similar with those parallel works on Smale spaces (c.f. [Put96] and references therein). Starting from a Smale space (X, φ), where X is a compact metric space and $\varphi\colon X \to X$ is a homeomorphism, one may "discretise" its dynamics using the *Markov partitions* to obtain a symbolic dynamical system (Σ, σ), which assigns to each $x \in X$ a binary code. This gives a Cantor dynamical system. From a sequence in (Σ, σ) viewed as the "binary expansion" of a number, we have natural equivalence relations thereamong. This generates a dynamical system $(\Sigma/\sim, \sigma/\sim)$ on the interval.

Putnam and Treviño used a similar idea to build their translation surfaces, displayed by the following diagram:

$$\boxed{\begin{array}{c}\text{Ordered bi-infinite}\\ \text{Bratteli diagram}\\ \mathcal{B}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Ordered path space}\\ X_{\mathcal{B}}\end{array}} \xRightarrow[\text{the order}]{\text{Gluing using}} \boxed{\begin{array}{c}\text{Translation surface}\\ S_{\mathcal{B}}\end{array}}$$

A cliffhanger is that, with this order one is able to find the "singularities" of the surface!

Throughout this lecture, we will be extentively using the ordered bi-infinite Bratteli diagram in Figure 2.1 to display examples. For each $n \in \mathbb{Z}$, V_n consists of a unique vertex v_n; and for each n, there are two direct paths from v_n to v_{n+1} in E_n labelled by 1 and 0 satisfying $1 \geq 0$. Note that we have "transposed" the diagram as opposed to the standard convention. This is helpful as we are concerning about bi-infinite Bratteli diagrams.

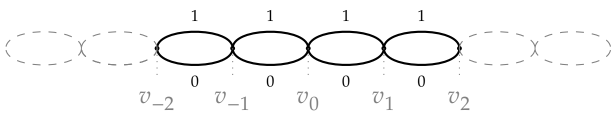


Figure 2.1: A bi-finite Bratteli diagram, used to display examples from this lecture

2.1 Topology and measures on the path space

From now on, we will assume that $\mathcal{B} = (V, E, r, s)$ is a bi-infinite Bratteli diagram.

Definition 2.1. A *state* on $\mathcal{B}$ is a pair of functions

$$(\nu_r, \nu_s), \qquad \nu_s, \nu_s\colon V \rightrightarrows [0, +\infty)$$

such that

$$\nu_r(v) = \sum_{e \in r^{-1}(v)} \nu_r(s(e)), \qquad \nu_s(v) = \sum_{e \in r^{-1}(v)} \nu_s(r(e))$$

hold for all $v \in V$.

We say a state ν is *normalised*, if $\sum_{v \in V_0} \nu_r(v)\nu_s(v) = 1$.

We say a state is *faithful*, if $\nu_r(v) > 0$ and $\nu_s(v) > 0$ for all $v \in V$.

Example 2.2. A normalised state on the diagram of Figure 2.1 is given by

$$\nu_r(v) = 2^n, \qquad \nu_s(v) = 2^{-n}$$

for all $n \in \mathbb{Z}$ and $v \in V_n$.

Easily one can prove the following "shift invariance" property of states:

Proposition 2.3. *Let (ν_r, ν_s) be a state on $\mathcal{B}$. Then for all $n \in \mathbb{Z}$:*

$$\sum_{v \in V_n} \nu_r(v)\nu_s(v) = \sum_{v \in V_0} \nu_r(v)\nu_s(v).$$

Proposition 2.4. *Every bi-infinite Bratteli diagram $\mathcal{B}$ possesses a state. If $\mathcal{B}$ is* simple *(Definiton 2.6), then every state is faithful.*

Recall that $X_{\mathcal{B}}$ denotes the (bi-infinite) path space of $\mathcal{B}$, whose elements are bi-infinite words

$$x = (\ldots, x_{i+1}, x_i, \ldots, x_2, x_1, \ldots) \quad \in \quad \prod_{i \in \mathbb{Z}} E_i,$$

such that $s(x_{i+1}) = r(x_i)$ for all i.

Definition 2.5. Let $v \in V_n$, define X_v^+ as the space of uni-infinite paths starting at v, whose elements consists of uni-infinite words

$$x = (\ldots, x_{i+1}, x_i, \ldots, x_{n+2}, x_{n+1}) \quad \in \quad \prod_{i \geq n+1} E_i$$

such that $s(x) := s(x_{n+1}) = v$.

Similarly we define X_v^- as the space of uni-infinite paths ending at v.

Definition 2.6 ([PT22, Definition 2.4, Lemma 3.3])**.** We say $\mathcal{B}$ is *simple*, if for every $m \in \mathbb{Z}$, there are integers l and n with $l < m < n$, such that:

(1) There is a path from every vertex of V_l to every vertex of V_m.
(2) There is a path from every vertex of V_m to every vertex of V_n.

We say $\mathcal{B}$ is *strongly simple*, if $\mathcal{B}$ is simple and $X_v^{\pm}$ are infinite for all $v \in V$. Due to the simplicity of $\mathcal{B}$, this is equivalent to "... for some $v \in V$".

Recall that $X_{\mathcal{B}}$ is equipped with the Tychonoff topology (product topology) of $\prod_{i \in \mathbb{Z}} E_i$. In particular, we have:

Proposition 2.7. *Equip $X_{\mathcal{B}} \subseteq \prod_{i \in \mathbb{Z}} E_i$ with the Tychonoff topology. Then:*

(1) *$X_{\mathcal{B}} \subseteq \prod_{i \in \mathbb{Z}} E_i$ is closed, hence a compact Hausdorff space.*
(2) *$X_{\mathcal{B}}$ is* totally disconnected*, whose clopen basis consists of subsets of the form*

$$X_{s(p)}^- p X_{r(p)}^+, \qquad p \text{ is a finite path.}$$

Those subsets are called cylinder sets *of $X_{\mathcal{B}}$.*

(3) *$X_{\mathcal{B}}$ carries an* ultrametric[3]*, given by*

$$d(x, y) := \inf_{n \geq 0} \left\{ 2^{-n} \,\middle|\, x_i = y_i \text{ for all } 1 - n \leq i \leq n \right\}.$$

(4) *If $\mathcal{B}$ is strongly simple, then $X_{\mathcal{B}}$ is Cantor (i.e. does not have isolated points).*

A state on $\mathcal{B}$ generates a (probability) measure on $X_{\mathcal{B}}$:

Proposition 2.8 ([PT22, Lemma 3.8])**.** *Let (ν_r, ν_s) be a state on $\mathcal{B}$. Then there is a unique probability measure $\nu_r \times \nu_s$ on $\mathcal{B}$ such that*

$$\nu_r \times \nu_s \,(X_{s(p)}^- p X_{r(p)}^+) = \nu_r(s(p))\nu_s(r(p)).$$

If (ν_r, ν_s) is faithful, then $\nu_r \times \nu_s$ has full support.

If $\mathcal{B}$ is strongly simple, then $\nu_r \times \nu_s$ has no atoms.

[3] An ultrametric is a metric d such that $d(x, y) \leq \max\{d(y, z), d(x, z)\}$ for all z.

2.2 Topology and measures on the leaves

Now we define the leaves, roughly as the tail equivalence classes of $X_{\mathcal{B}}$. This is, however, not precise, because those are not connected.

Definition 2.9. Let $x \in X_{\mathcal{B}}$ and $N \in \mathbb{Z}$. Define

$$T_N^+(x) := \{y \in X_{\mathcal{B}} \mid y_n = x_n \text{ for all } n > N\},$$
$$T_N^-(x) := \{y \in X_{\mathcal{B}} \mid y_n = x_n \text{ for all } n \leq N\}.$$

The set of right (resp. left) tail equivalence classes is

$$T^+(x) := \bigcup_{N \in \mathbb{Z}} T_N^+(x), \quad \text{resp.} \quad T^-(x) := \bigcup_{N \in \mathbb{Z}} T_N^-(x).$$

Note that $T^\pm(x)$ are subsets of $X_{\mathcal{B}}$. However, the density of orbits (equivalence classes) in $X_{\mathcal{B}}$ makes their subspace topologies trivial and hence not useful for our purpose. The correct topology is the *inductive limit topology*: We equip $T_N^\pm(x)$ with the relative topology on $X_{\mathcal{B}}$ and equip $T^\pm(x)$ with their inductive limit topology. Recall that this means $A \subseteq T^\pm(x)$ is open iff $A \cap T_N^\pm(x)$ is open for all N.

This is similar with equipping $\mathbb{R} = \bigcup_{n \in \mathbb{N}}(-n, n)$ with the inductive limit topology coming from the natural topologies on intervals. In that case we recovers the standard topology on $\mathbb{R}$.

The following proposition is the one-side version of Proposition 2.8:

Proposition 2.10 ([PT22, Proposition 3.9])**.** *Let $x = (\dots, x_{i+1}, x_i, \dots) \in X_{\mathcal{B}}$. Then there exists a measure ν_r^x on $T^+(x)$ such that*

$$\nu_r^x(X_{s(p)}^- p x_{(n,+\infty)}) = \nu_r(s(p_{m+1}))$$

for all $p \in E_{m,n}$ satisfying $r(p) = r(x_n)$, where

$$x_{(n,+\infty)} := (\dots, x_{n+2}, x_{n+1}) \in \prod_{i>n} E_i,$$
$$E_{m,n} := \prod_{m<i\leq n} E_i = \{\text{Finite paths from } V_m \text{ to } V_n\}.$$

Similarly, there exists a measure ν_s^x on $T^-(x)$ such that

$$\nu_s^x(x_{(-\infty,m]} p X_{r(p)}^+) = \nu_s(r(p_n))$$

for all $p \in E_{m,n}$ satisfying $s(p) = s(x_m)$.

If (ν_r, ν_s) is faithful, then ν_r^x and ν_s^x have full support. If $\mathcal{B}$ is strongly simple, then ν_r^x and ν_s^x have no atoms.

2.3 Orders on the path space

2.3.1 On finite paths

Recall that (Definition 1.12) an *ordered* bi-infinite Bratteli diagram is a bi-infinite Bratteli diagram $\mathcal{B} = (V, E, r, s)$ together with partial orders $\leq_s, \leq_r$ on E, such that for every pair of edges $e, e' \in E$:

- e and e' are $\leq_s$ comparable, if $s(e) = s(e')$.
- e and e' are $\leq_r$ comparable, if $r(e) = r(e')$.

This extends to a partial order on all *finite* paths using the *lexicographic order*.

Example 2.11. Consider the two finite paths ≀ and ≀ in Figure 2.2. They have the same starting and ending points and hence comparable in both $\leq_r$ and $\leq_s$ using the lexicographic order, which can be viewed as comparing two decimals in the binary expansion. In the $\leq_s$-comparison of two paths we look for the first different edge in those paths from the *left*, then read "from left to right". Then the comparison reads

$$0011 \leq_s 1001.$$

The $\leq_r$-comparison is from the opposite direction, namely compare the first different edge from the *right*. Thus

$$0011 \geq_r 1001.$$

(a) ≀=0011 (b) ≀=1001

Figure 2.2: "Binary expansions" of two finite paths ≀ and ≀

Definition 2.12. Let X be a linearly ordered set and $x, y \in X$. We say y is the *successor* of x and x is the *predecessor* of y if $x < y$ and there is no z such that $x < z < y$.

For every edge $e \in E$, since $s^{-1}(s(e))$ is linearly ordered in $\leq_s$ and $r^{-1}(r(e))$ is linearly ordered in $\leq_r$, we are able to talk about the *successor* or *predecessor* of e in $\leq_r$ or $\leq_s$, providing those exist. These notions can be extended to finite paths as well.

Example 2.13. In Figure 2.3, ≀ has $\leq_s$-successor ≀, and ≀ has $\leq_s$-predecessor ≀.

(a) ≀=0000 (b) ≀=0001

Figure 2.3: ≀ is the $\leq_s$-successor of ≀

2.3.2 On infinite paths

Definition and Lemma 2.14 ([PT22, Proposition 4.1])**.** *$\mathcal{B}$ contains an infinite path, such that every edge of it is s-maximal (resp. r-maximal, resp. s-minimal, resp. r-minimal). We denote the collection of such paths by $X_{\mathcal{B}}^{s\text{-}max}$ (resp. $X_{\mathcal{B}}^{r\text{-}max}$, resp. $X_{\mathcal{B}}^{s\text{-}min}$, resp. $X_{\mathcal{B}}^{r\text{-}min}$). Each of them is closed in $X_{\mathcal{B}}$.*

If there exists some K such that $\#V_n \leq K$ for all $n \in \mathbb{Z}$. Then the cardinal of each of these sets is at most K.

We define

$$X_{\mathcal{B}}^{ext} := X_{\mathcal{B}}^{s\text{-}max} \cup X_{\mathcal{B}}^{r\text{-}max} \cup X_{\mathcal{B}}^{s\text{-}min} \cup X_{\mathcal{B}}^{r\text{-}min}.$$

Recall the partial orders $\leq_s$ and $\leq_r$ on $X_{\mathcal{B}}$ (Definition 1.13): given infinite paths $x = (x_i)_i$ and $y = (y_i)_i$, we say:

- $x \leq_r y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \geq n$ and $x_n \leq_r y_n$.
- $x \leq_s y$, if there exists $n \in \mathbb{Z}$, such that $x_i = y_i$ for all $i \leq n$ and $x_n \leq_s y_n$.

So x and y are comparable in $\leq_r$ (resp. $\leq_s$) iff they are left (resp. right) tail equivalent. In particular, this means that $T^-(x)$ (resp. $T^+(x)$) is linearly $\leq_s$-ordered (resp. $\leq_r$-ordered). Thus for infinite paths that are in the same tail equivalence class, we are able to compare them and speak about their successors and predecessors as well providing they exist.

2.4 Defining the surface

From now on, we assume that $\mathcal{B}$ is strongly simple.

Definition 2.15. Define the *source boundary* $\partial_s X_{\mathcal{B}}$ of $X_{\mathcal{B}}$ as

$$\partial_s X_{\mathcal{B}} := \{x \in X_{\mathcal{B}} \mid x \text{ has either an } s\text{-successor or an } s\text{-predecessor}\}.$$

Remark 2.16. We claim (without proving here) that an infinite path cannot have both an s-successor or an s-predecessor. A good example is to consider the decimal expansion of reals, which can be viewed as a quotient of a Cantor denary system: the decimal

$$0.99999\cdots$$

has a successor $1.0000\cdots$. They are not yet equivalent in the Cantor set, but "identified" and hence becoming representatives of the same *real number*. However, $0.999\cdots$ does not have a predecessor.

Definition 2.17. Define

$$\Delta_s \colon \partial_s X_{\mathcal{B}} \to \partial_s X_{\mathcal{B}}$$

by sending x to its s-successor or s-predecessor. The previous remark guarantees that this map is indeed well-defined.

Clearly we have $\Delta_s^2 = \mathrm{id}$. Similarly we define $\partial_r X_{\mathcal{B}}$ and $\Delta_r X_{\mathcal{B}}$. Notice that $\Delta_r(x)$ (resp. $\Delta_s(x)$) and x are right (resp. left) tail equivalent.

Definition and Lemma 2.18. *Define the boundary of $X_{\mathcal{B}}$ as*

$$\partial X_{\mathcal{B}} := \partial_s X_{\mathcal{B}} \cap \partial_r X_{\mathcal{B}}.$$

Then Δ_s and Δ_r leave $\partial X_{\mathcal{B}}$ invariant. The set of singular points *of $X_{\mathcal{B}}$ is defined as*

$$\Sigma_{\mathcal{B}} := \{x \in \partial X_{\mathcal{B}} \mid \Delta_r \circ \Delta_s(x) \neq \Delta_s \circ \Delta_r(x)\}.$$

Why is such a point "singular"? Well, as we shall anticipate, gluing the path space $X_{\mathcal{B}}$ using the order yields a flat surface. But the condition $\Delta_r \circ \Delta_s(x) \neq \Delta_s \circ \Delta_r(x)$ says that the surface is not "flat" around x.

Example 2.19. Figure 2.4 displays an example of a singular point $\wr$. In the "binary expansion" it can be written as $\cdots 00100 \cdots$, which belongs to the boundary of $X_{\mathcal{B}}$. But:

$$\cdots 00100 \cdots \xrightarrow{\Delta_s} \cdots 00011 \cdots \xrightarrow{\Delta_r} \cdots 11101 \cdots;$$
$$\cdots 00100 \cdots \xrightarrow{\Delta_r} \cdots 11000 \cdots \xrightarrow{\Delta_s} \cdots 10111 \cdots.$$

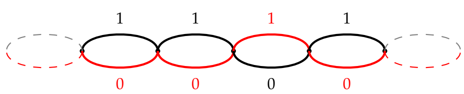


Figure 2.4: Example of a singular point $\wr = \cdots 00100 \cdots$

Definition and Lemma 2.20. *Define*

$$Y_{\mathcal{B}} := X_{\mathcal{B}} \setminus (X_{\mathcal{B}}^{ext} \cup \Sigma_{\mathcal{B}}).$$

Both $X_{\mathcal{B}}^{ext}$ and $\Sigma_{\mathcal{B}}$ are countable, closed subsets of $X_{\mathcal{B}}$. So $Y_{\mathcal{B}}$ is open.

The surface $S_{\mathcal{B}}$ is

$$S_{\mathcal{B}} := Y_{\mathcal{B}}/\sim$$

with the equivalence relation $\sim$ generated by:

$$\begin{aligned} y &\sim \Delta_s(y), && \text{if } y \in \partial_s X_{\mathcal{B}} \cap Y_{\mathcal{B}}; \\ y &\sim \Delta_r(y), && \text{if } y \in \partial_r X_{\mathcal{B}} \cap Y_{\mathcal{B}}. \end{aligned}$$

3 Translation surfaces, groupoids and C*-algebras

3.1 Translation surfaces

Let $\mathcal{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram, with orders $\leq_s$ and $\leq_r$.

Recall from Dimitris's talk that we have removed singular and boundary points of $X_{\mathcal{B}}$ to get $Y_{\mathcal{B}}$. The translation surface $S_{\mathcal{B}}$ is obtained by gluing certain pairs of points in $Y_{\mathcal{B}}$, using the orders of $\mathcal{B}$ and $X_{\mathcal{B}}$, see Definition and Lemma 2.20. We explain this construction in more detail.

We introduce first more symbols and notations. For our convenience, a complete list of symbols used in [PT22] is provided in the notes.

Definition 3.1. Define

$$S^s_{\mathcal{B}} := Y_{\mathcal{B}} \Big/ x \sim \Delta_s(x),\ x \in Y_{\mathcal{B}} \cap \partial_s X_{\mathcal{B}};$$

$$S^r_{\mathcal{B}} := Y_{\mathcal{B}} \Big/ x \sim \Delta_r(x),\ x \in Y_{\mathcal{B}} \cap \partial_r X_{\mathcal{B}}.$$

Then we have an commuting diagram consisting of $X_{\mathcal{B}}$, $Y_{\mathcal{B}}$, $S^s_{\mathcal{B}}$, $S^r_{\mathcal{B}}$, $S_{\mathcal{B}}$ with obvious quotient maps between them:

$$\begin{tikzcd}
 & X_{\mathcal{B}} & \\
 & Y_{\mathcal{B}} \arrow[u, hook] \arrow[dl, "\pi^r"'] \arrow[dr, "\pi^s"] \arrow[dd, "\pi"] & \\
S^r_{\mathcal{B}} \arrow[dr, "\rho^s"'] & & S^s_{\mathcal{B}} \arrow[dl, "\rho^r"] \\
 & S_{\mathcal{B}} &
\end{tikzcd}$$

To finalise the construction of a translation surface $S_{\mathcal{B}}$, we must construct translation atlas for it.

Definition 3.2 ([PT22, Definition 6.1])**.** Denote by $E^Y_{m,n}$ the collection of finite paths $p \in E_{m,n}$ with:

- p is neither s-maximal, s-minimal, r-maximal nor r-minimal.
- $X^-_{s(p)} p X^+_{r(p)} \subseteq Y_{\mathcal{B}}$.

Roughly speaking, $E^Y_{m,n}$ is just the set of finite paths $p \in E_{m,n}$ which extend to infinite paths in $Y_{\mathcal{B}}$.

Definition 3.3 ([PT22, Definition 6.9])**.** Denote by $E^{s/r}_{m,n}$ the set of quadruples

$$p = (p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}), \qquad p_{i,j} \in E^Y_{m,n},$$

such that:

- $p_{i,j+1}$ is an s-successor of $p_{i,j}$.
- $p_{i+1,j}$ is an r-successor of $p_{i,j}$.

Namely, we have the following diagram of successors:

$$\begin{tikzcd}
p_{1,1} \arrow[r, "s\text{-succ.}"] \arrow[d, "r\text{-succ.}"'] & p_{1,2} \arrow[d, "r\text{-succ.}"] \\
p_{2,1} \arrow[r, "s\text{-succ.}"'] & p_{2,2}
\end{tikzcd}$$

Definition 3.4 ([PT22, Definition 6.9, Theorem 6.13])**.** Define

$$V_{i,j}(p) := \left(X^-_{s(p_{i,j})} \setminus X^{r\text{-}m_i}_{s(p_{i,j})}\right) p_{i,j} \left(X^+_{r(p_{i,j})} \setminus X^{s\text{-}m_j}_{r(p_{i,j})}\right)$$

where $i, j \in \{1, 2\}$, $m_1 = \min$, $m_2 = \max$.

Define

$$V(p) := \bigcup_{i,j} V_{i,j}(p) \subseteq Y_{\mathcal{B}}, \qquad Y(p) := \pi(V(p)) \subseteq S_{\mathcal{B}}.$$

Example 3.5. Figure 3.1 gives an example of $V_{1,1}$ where $p_{1,1}$ is the path $\wr$ of length 1 connecting v_0 and v_1 labelled by 1. Then

$$V_{1,1}(p) := \left(X^-_{v_0} \setminus X^{r\text{-min}}_{v_0}\right) p_{1,1} \left(X^+_{v_1} \setminus X^{s\text{-min}}_{v_1}\right).$$

It consists of (bi-infinite) paths x in $\mathcal{B}$, satisfying:

(1) x contains the path $p_{1,1} = \wr$ as a segment.

(2) x does *not* contain the minimal path ending at v_0: namely, the dashed path $\wr = \cdots 000$ which ends at v_0.

(3) x does *not* contain the minimal path starting from v_0: namely, the dashed path $\wr = 000\cdots$ which starts from v_0.

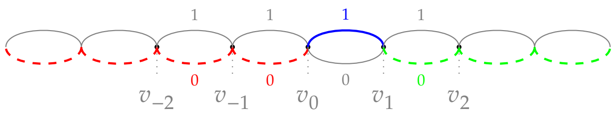


Figure 3.1: Example of $V_{1,1}$

We will see that $V(p)$'s are the chart domains for the translation atlas of $S_{\mathcal{B}}$. To construct the charts, we need the following

Definition and Lemma 3.6 ([PT22, Proposition 3.7, Lemma 3.8])**.** *Let (ν_s, ν_r) be a normalised state on $\mathcal{B}$ (see Definition 2.1). Denote by $X^+_{\mathcal{B}}$ (resp. $X^-_{\mathcal{B}}$) the set of right-infinite (resp. left-infinite) paths of $\mathcal{B}$. There there is a unique measure ν_s on $X^+_{\mathcal{B}}$ such that*

$$\nu_s(pX^+_{r(p)}) = \nu_s(r(p))$$

and a unique measure ν_r on $X^-_{\mathcal{B}}$ such that

$$\nu_r(X^-_{s(p)}p) = \nu_r(s(p))$$

for any finite path p.

This is proved by truncating the bi-infinite Bratteli diagram $\mathcal{B}$ into a Bratteli diagram which is only left or right infinite, then using [PT22, Proposition 3.7]. Note that the measure in Proposition 2.8 is a product measure of ν_r and ν_s defined here.

Definition 3.7 ([PT22, Definition 4.7])**.** Let $v \in V$. Define

$$\begin{aligned} \varphi^v_s \colon X^+_v &\to [0, \nu_s(v)], \qquad x \mapsto \nu_s(\{y \in X^+_v \mid y \leq_s x\}), \\ \varphi^v_r \colon X^-_v &\to [0, \nu_r(v)], \qquad x \mapsto \nu_r(\{y \in X^-_v \mid y \leq_r x\}). \end{aligned}$$

3.1.1 The translation atlas

Now we are able to describe the translation atlas for $S_{\mathcal{B}}$. We assume for now the following **standing assumptions**:

- $\mathcal{B}$ has *finite rank*, that is, V_n is uniformly bounded.
- $\mathcal{B}$ is *strongly simple*, that is, $\mathcal{B}$ is simple and $\#X_v^{\pm} = \infty$ for every $v \in V$. (See Definition 2.6).
- $X_{\mathcal{B}}^{\text{ext}} \cap \partial_r X_{\mathcal{B}} = \emptyset$ and $X_{\mathcal{B}}^{\text{ext}} \cap \partial_s X_{\mathcal{B}} = \emptyset$.

Definition and Lemma 3.8 ([PT22, Definition 6.9, Theorem 6.13]). *Let $p \in E_{m,n}^{r/s}$. Define $V(p) \subseteq Y_{\mathcal{B}}$ and $Y(p) = \pi(V(p)) \subseteq S_{\mathcal{B}}$ as in Definition 3.4.*

- *We define $\psi^p : V(p) \longrightarrow \mathbb{R}^2$ by*

$$\psi^p(x) := \left(\varphi_r^{s(x)}(x_{(-\infty,m]}), \varphi_s^{r(x)}(x_{[n,+\infty)})\right) + \begin{cases} (-\nu_r(s(p_{1,1})), -\nu_s(r(p_{1,1}))) & x \in V_{1,1}(p), \\ (-\nu_r(s(p_{1,1})), 0) & x \in V_{1,2}(p), \\ (0, -\nu_s(r(p_{1,1}))) & x \in V_{2,1}(p), \\ (0,0) & x \in V_{2,2}(p). \end{cases}$$

- *There is a unique map $\eta^p : Y(p) \longrightarrow \mathbb{R}^2$ satisfying $\eta^p = \psi^p \circ \pi$.*

Proposition 3.9. *Define ϕ^p, η^p as in Definition and Lemma 3.8. We have the following properties:*

(1) $\bigcup_{n\in\mathbb{N}}\{V(p)\}_{p\in E_{-n,n}^{r/s}}$ *is an open over of $Y_{\mathcal{B}}$.*

(2) *$V(p)$ is invariant under Δ_r and Δ_s.*

(3) *ψ^p is continuous.*

(4) *$\psi^p(x) = \psi^p(y)$ iff $\pi(x) = \pi(y)$.*

(5) *For any $p \in E_{-m,m}^{r/s}$ and $q \in E_{-n,n}^{r/s}$, there exists $c_{p,q} \in \mathbb{R}^2$ such that*

$$\psi^p(x) = \psi^q(x) - c_{p,q}, \qquad \text{for all } x \in V(p) \cap V(q).$$

Finally, we are able to describe the translation atlas for $S_{\mathcal{B}}$:

Theorem 3.10 ([PT22, Theorem 6.13]). *There exists a sequence of natural numbers $\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$, such that*

$$\left\{ \eta^p : Y_p \longrightarrow \mathbb{R}^2 \,\middle|\, p \in \bigcup_{k\geq 1} E_{-n_k,n_k}^{r/s} \right\}$$

is a translation atlas for $S_{\mathcal{B}}$.

3.2 Groupoids

In the following, we will introduce several groupoids, and construct Haar systems thereon to pass to their (reduced) C*-algebras. Almost all of these groupoids are defined by *equivalence relations*. That is, let X be a topological space, and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X, equipped with the subspace topology. Then

$$\mathcal{R} \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} X$$

is a topological groupoid, with multiplication $(x,y)\cdot(y,z) := (x,z)$ and inverse $(x,y)^{-1} := (y,x)$. As a special case, the pair groupoid is defined by the full equivalence relation $\mathcal{R} := X \times X$.

The groupoids introduced in this section (and in the paper [PT22]) as well as their relations are described by the diagram below:

$$\begin{tikzcd}
 & T^+(X_{\mathcal{B}}) & & \\
 & T^+(Y_{\mathcal{B}}) \arrow[u, hook, "1"] \arrow[dl, "\pi^r\times\pi^r"'] & T^\sharp(Y_{\mathcal{B}}) \arrow[l, hook', "2"'] \arrow[dr, "\pi^s\times\pi^s"] & \\
T^+(S^r_{\mathcal{B}}) & & & T^\sharp(S^s_{\mathcal{B}}) \arrow[dl, "\rho^r\times\rho^r"] \\
 & & T^\sharp(S_{\mathcal{B}}) & \\
 & & \mathcal{F}^+_{\mathcal{B}} \arrow[u, hook, "3"'] &
\end{tikzcd} \tag{3.11}$$

where:

(1) $T^+(Y_{\mathcal{B}}) := T^+(X_{\mathcal{B}})\big|_{Y_{\mathcal{B}}}^{Y_{\mathcal{B}}}$.

(2) $T^\sharp(Y_{\mathcal{B}}) := \left\{(x,y) \in T^+(Y_{\mathcal{B}}) \,\middle|\, \text{If } x,y \in \partial_s X_{\mathcal{B}}, \text{ then } (\Delta_s(x), \Delta_s(y)) \in T^+(Y_{\mathcal{B}})\right\}$.

(3) $\mathcal{F}^+_{\mathcal{B}} := \left\{(x,y) \in T^\sharp(S_{\mathcal{B}}) \,\middle|\, x, y \text{ are in the same connected component}\right\}$.

By $\mathcal{G} \hookrightarrow \mathcal{H}$ I mean $\mathcal{G}$ is an open subgroupoid[4] of $\mathcal{H}$. The symbol $\sim$ and note "stably equivalent" on these arrows are not about the groupoids themselves, but rather indicate the relation between their C*-algebras, as we will see in the next section.

$T^+(X_{\mathcal{B}})$ and $T^+(Y_{\mathcal{B}})$. The right tail equivalence relation on $X_{\mathcal{B}}$ generates a groupoid, which we denoted by $T^+(X_{\mathcal{B}})$. It carries the inductive limit topology from $T^+_N(X_{\mathcal{B}})$, the latter defined by "right tail equivalence starting from V_N". With this topology, $T^+(X_{\mathcal{B}})$ is locally compact and Hausdorff.

The Haar system on $T^+(X_{\mathcal{B}})$ is $\{\nu_r^x\}_{x\in X_{\mathcal{B}}}$ as described in Proposition 2.10.

Restricting $T^+(X_{\mathcal{B}})$ to $Y_{\mathcal{B}} \subseteq X_{\mathcal{B}}$ gives a new groupoid $T^+(Y_{\mathcal{B}})$, which can be equipped with the same Haar system with $T^+(X_{\mathcal{B}})$ because $Y_{\mathcal{B}}$ is obtained by removing countably many points from $X_{\mathcal{B}}$, and ν_r^x is not atomic.

$T^+(S^r_{\mathcal{B}})$. Why not $T^+(S^s_{\mathcal{B}})$? The gluing $\pi^r : Y_{\mathcal{B}} \to S^r_{\mathcal{B}}$ identifies x with $\Delta_r(x)$ in $Y_{\mathcal{B}}$. But note that x and $\Delta_r(x)$ are already right tail equivalent. That says, we "do not obtain very much new information" from this process.

We define
$$T^+(S^r_{\mathcal{B}}) := \pi^r \times \pi^r (T^+(Y_{\mathcal{B}}))$$
and equip it with the quotient topology. $\pi^r \times \pi^r$ is a continuous proper surjection, thus it gives a Haar system $\{\pi^r_* \nu_r^x\}_{x\in S^r_{\mathcal{B}}}$.

The gluing $\pi^s : Y_{\mathcal{B}} \to S^s_{\mathcal{B}}$ is however different, because x and $\Delta_s(x)$ are not right tail equivalent. Thus we cannot define $T^+(X^s_{\mathcal{B}})$ in a similar naïve way. We have to pass to a subgroupoid of $T^+(Y_{\mathcal{B}})$.

[4] A subgroupoid of a groupoid $\mathcal{G}$ is a subset $\mathcal{H} \subseteq \mathcal{G}$ such that, the structure maps of $\mathcal{G}$ restricted to $\mathcal{H}$ turns it into a groupoid.

$T^\sharp(Y_{\mathcal{B}})$. The correct construction to encode the equivalence generated by Δ_s is given by the following open subgroupoid of $T^+(Y_{\mathcal{B}})$:

$$T^\sharp(Y_{\mathcal{B}}) := \left\{(x,y) \in T^+(Y_{\mathcal{B}}) \mid \text{If } x, y \in \partial_s X_{\mathcal{B}}, \text{ then } (\Delta_s(x), \Delta_s(y)) \in T^+(Y_{\mathcal{B}})\right\}.$$

As an open subgroupoid, restricting the Haar system $\{\nu_r^x\}_{x \in Y_{\mathcal{B}}}$ to it yields a Haar system.

Remark 3.12 ([PT22, Proposition 7.6]). Note that $T^\sharp(Y_{\mathcal{B}}) = T^+(Y_{\mathcal{B}})$ if

$$\#(\text{tail equivalence classes of } s\text{-min paths}) = \#(\text{tail equivalence classes of } s\text{-max paths}) = 1.$$

$T^\sharp(S^s_{\mathcal{B}})$. Now the construction mimicks that of $T^+(S^r_{\mathcal{B}})$, except that we are working with $T^\sharp(Y_{\mathcal{B}})$. We define

$$T^\sharp(S^s_{\mathcal{B}}) := \pi^s \times \pi^s(T^\sharp(Y_{\mathcal{B}}))$$

and equip it with the quotient topology. $\pi^s \times \pi^s$ is a continuous proper surjection, thus it gives a Haar system $\{\pi^s_* \nu_r^x\}_{x \in S^s_{\mathcal{B}}}$.

$T^\sharp(S_{\mathcal{B}})$. Define

$$T^\sharp(S_{\mathcal{B}}) := \rho^r \times \rho^r(T^\sharp(X^s_{\mathcal{B}})) = \pi \times \pi(T^\sharp(Y_{\mathcal{B}})).$$

with the Haar system $\{\rho^r_* \pi^s_* \nu_r^x\}_{x \in S_{\mathcal{B}}}$

$\mathcal{F}^+_{\mathcal{B}}$. $T^\sharp(S_{\mathcal{B}})$ is almost the foliation groupoid, except that its leaves may not be connected. The last step is to replace the leaves by its connected components:

$$\mathcal{F}^+_{\mathcal{B}} := \left\{(x,y) \in T^\sharp(S_{\mathcal{B}}) \mid x, y \text{ are in the same connected component}\right\}.$$

I should leave its topology and Haar system as an exercise.

3.3 C*-algebras

The groupoids from the previous section yield groupoid C*-algebras, whose relations and symbols are shown in the diagram below:

$$\begin{tikzcd}
& A^+_{\mathcal{B}} := \mathrm{C}^*(T^+(X_{\mathcal{B}})) & & \\
& A^{Y+}_{\mathcal{B}} := \mathrm{C}^*(T^+(Y_{\mathcal{B}})) \arrow[u, hook, "\text{hereditary}"] \arrow[dl, "\sim"'] & \mathrm{C}^*(T^\sharp(Y_{\mathcal{B}})) \arrow[l, hook'] & \\
\mathrm{C}^*(T^+(S^r_{\mathcal{B}})) & & & B^+_{\mathcal{B}} := \mathrm{C}^*(T^\sharp(S^s_{\mathcal{B}})) \arrow[ul, hook'] \arrow[dl, "\sim"] \\
& & \mathrm{C}^*(T^\sharp(S_{\mathcal{B}})) & \\
& & C^+_{\mathcal{B}} := \mathrm{C}^*(\mathcal{F}^+_{\mathcal{B}}) \arrow[u, hook] &
\end{tikzcd} \tag{3.13}$$

Their K-theory will be discussed thoroughly in the coming talk by Yufan. Many of these C*-algebras have the same K-theory because they are stably isomorphic or even isomorphic. Besides those, we will also define several inductive systems of C*-algebras:

$$(A^+_{m,n}), (AC^+_{m,n}); (A^{Y+}_{m,n}), (AC^{Y+}_{m,n}); (B^+_{m,n}); (C^+_{m,n}).$$

The first one consists of finite-dimensional C*-algebras, and their union is dense in $A^+_{\mathcal{B}} := \mathrm{C}^*(T^+(X_{\mathcal{B}}))$. This realises $A^+_{\mathcal{B}}$ as an AF-algebra. The other inductive system provides us with useful short exact sequences, allowing us to compute the K-theory of the foliation C*-algebra $C^+_{\mathcal{B}} := \mathrm{C}^*(\mathcal{F}^+_{\mathcal{B}})$.

3.3.1 $A^+_{\mathcal{B}}$, $A^{Y+}_{\mathcal{B}}$, the inductive systems $(A^+_{m,n})$, $(A^{Y+}_{m,n})$, $(AC^+_{m,n})$ and $(AC^{Y+}_{m,n})$

Definition 3.14. We define

$$A^+_{\mathcal{B}} := \mathrm{C}^*(T^+(X_{\mathcal{B}})), \qquad A^{Y+}_{\mathcal{B}} := \mathrm{C}^*(T^+(Y_{\mathcal{B}})).$$

We are going to construct an inductive system $(A^+_{m,n})_{m,n\in\mathbb{Z}}$, each $A_{m,n}$ being a finite-dimensional C*-algebra, and such that

$$A^+_{\mathcal{B}} = \overline{\bigcup_{m,n} A^+_{m,n}}.$$

This realises $A^+_{\mathcal{B}}$ as an AF-algebra.

Before doing that, we claim that the C*-subalgebra $A^{Y+}_{\mathcal{B}}$ is actually stably equivalent to $A^+_{\mathcal{B}}$:

Proposition 3.15. *$A^{Y+}_{\mathcal{B}}$ is a full hereditary* C*-*subalgebra of $A^+_{\mathcal{B}}$*[5]*, hence the inclusion $A^{Y+}_{\mathcal{B}} \hookrightarrow A^+_{\mathcal{B}}$ induces a stable equivalence of* C*-*algebras (see [Bro77]):*

$$A^{Y+}_{\mathcal{B}} \otimes \mathbb{K} \simeq A^+_{\mathcal{B}} \otimes \mathbb{K},$$

and an isomorphism in K*-theory.*

This is roughly because we only remove a countable set of points from $X_{\mathcal{B}}$ to obtain $Y_{\mathcal{B}}$. As a consequence, passing from $A^+_{\mathcal{B}}$ to $A^{Y+}_{\mathcal{B}}$ is quite minor.

Definition 3.16. Let $p, q \in E_{m,n}$. Define the map $a_{p,q} : T^+(X_{\mathcal{B}}) \to \mathbb{R}$ by:

$$a_{p,q}(x,y) := \begin{cases} \nu_r(s(p))^{-1/2}\nu_r(s(q))^{-1/2} & \text{if } x_{(m,n]} = p,\ y_{(m,n]} = q,\ x_{(n,+\infty)} = y_{(n,+\infty)}; \\ 0 & \text{otherwise.} \end{cases}$$

The definition of $a_{p,q}$ reveals them as "propagation kernels" on $T^+(X_{\mathcal{B}})$. This can be seen more clearly in [PT22, Proposition 8.2].

Definition 3.17. Let $v \in V$. Define

$$A^+_{m,n,v} := \operatorname{span}\left\{a_{p,q} \mid p, q \in E_{m,n},\ r(p) = r(q) = v\right\},$$

and

$$A^+_{m,n} := \bigoplus_{v\in V_n} A^+_{m,n,v}.$$

Proposition 3.18 ([PT22, Proposition 8.3])**.** *Define $A^+_{m,n,v}$ and $A^+_{m,n}$ as above. Then we have:*

[5]Let A be a C*-algebra. A C*-subalgebra $B \subseteq A$ is called *hereditary*, if for every $a \in A$ and $b \in B$, $0 \le a \le b$ implies $a \in B$. A hereditary C*-subalgebra is *full* if it is not contained in any proper closed ideal of A.

(1) $A^+_{m,n,v} \simeq \mathbb{M}_{j(m,n,v)}(\mathbb{C})$, *where*

$$j(m,n,v) := \#(\text{paths in } E_{m,n} \text{ with range } v).$$

(2) *As a corollary, $A^+_{m,n}$ is a finite-dimensional* C*-*algebra.*

(3) $A^+_{m-1,n} \subseteq A^+_{m,n} \subseteq A^+_{m,n+1}$. *So $(A^+_{m,n})$ is an inductive system.*

(4) $A^+_{\mathcal{B}}$ *is an AF-algebra given by the inductive system $(A^+_{m,n})$, namely*

$$A^+_{\mathcal{B}} = \overline{\bigcup_{m,n} A^+_{m,n}}.$$

The inductive system $(A^+_{m,n})$ for $A^+_{\mathcal{B}}$ can be "restricted to" $A^{Y+}_{\mathcal{B}}$ as well:

Definition and Lemma 3.19 ([PT22, Proposition 8.7]). *Define*

$$A^{Y+}_{m,n,v} := \operatorname{span}\left\{a_{p,q} \mid p,q \in E^Y_{m,n},\ r(p) = r(q) = v\right\},$$

where $a_{p,q}$ are as in Definition 3.16, and

$$A^{Y+}_{m,n} := \bigoplus_{v \in V_n} A^{Y+}_{m,n,v}.$$

Then $(A^{Y+}_{m,n})$ is an inductive system for $A^{Y+}_{\mathcal{B}}$. Moreover, all conditions of Proposition 3.18 hold after replacing "+" by "Y+".

Now we construct inductive systems $(AC^+_{m,n})$ and $(AC^{Y+}_{m,n})$, which consists of infinite-dimensional C*-algebras. Nevertheless, this inductive system will be useful for the K-theory computation.

Definition 3.20. Let $v \in V$. Define

$$AC^+_{m,n,v} := A^+_{m,n,v} \otimes \mathrm{C}(X^+_v),$$

and

$$AC^+_{m,n} := \bigoplus_{v \in V_n} AC^+_{m,n,v}.$$

Similarly we define $AC^{Y+}_{m,n,v}$ and $AC^{Y+}_{m,n}$ by replacing every "+" with "Y+" in every $A^+_{m,n,v}$ and $A^+_{m,n}$.

Remark 3.21 ([PT22, Proposition 8.4]). An equivalent definition of $AC^+_{m,n,v}$ is by

$$AC^+_{m,n,v} := \operatorname{span}\left\{a_{pp',qp'} \mid p,q \in E_{m,n},\ p' \in E_{n,n'},\ r(p) = r(q) = s(p') = v\right\},$$

where pp' and qp' are the concatenation of paths in the usual sense, and $a_{pp',qp'}$ is as in Definition 3.16. The map

$$a_{pp',qp'} \mapsto a_{p,q} \otimes \chi_{p'X^+_{r(p')}}$$

extends to an isomorphism $AC^+_{m,n,v} \simeq A^+_{m,n,v} \otimes \mathrm{C}(X^+_v)$.

3.3.2 $B^+_{\mathcal{B}}$, the inductive system $(B^+_{-n,n})$

Definition 3.22. Define

$$B^+_{\mathcal{B}} := \mathrm{C}^*(T^\sharp(S^s_{\mathcal{B}})).$$

There are inclusions of C*-algebra:

$$B^+_{\mathcal{B}} \subseteq \mathrm{C}^*(T^\sharp(Y_{\mathcal{B}})) \subseteq A^{Y+}_{\mathcal{B}} \subseteq A^+_{\mathcal{B}}.$$

We will display $B^+_{\mathcal{B}}$ as an inductive limit of C*-algebras $(B^+_{-n,n})$ which are not finite-dimensional. Note that a C*-subalgebra of an AF-algebra is typically not AF. So $B^+_{\mathcal{B}}$, as a C*-subalgebra of the AF-algebra $A^{Y+}_{\mathcal{B}}$, need not be AF.

Definition 3.23. Define

$$B^+_{m,n} := AC^{Y+}_{m,n} \cap B^+_{\mathcal{B}}.$$

Then every $B^+_{m,n}$ $(m < n)$ is a C*-subalgebra of $B^+_{\mathcal{B}}$.

Proposition 3.24 ([PT22, Theorem 8.9])**.** *Define $B^+_{m,n}$ as above. Then:*

(1) $B^+_{m,n} \subseteq B^+_{m-1,n+1}$. *So $(B^+_{-n,n})$ is an inductive system.*

(2) $B^+_{\mathcal{B}}$ *is the inductive limit* C*-*algebra given by the inductive system, that is,*

$$B^+_{\mathcal{B}} = \overline{\bigcup_n B^+_{-n,n}}.$$

The K-theory of $B^+_{m,n}$ can be computed by a short exact sequence. For this, we must introduce another inductive system of groupoids.

Definition and Lemma 3.25 ([PT22, Definition 6.4, Page 43])**.** *Denote by $E^s_{m,n}$ the set of pairs*

$$p = (p_1, p_2), \qquad p_i \in E^Y_{m,n},$$

such that p_2 is an s-successor of p_1.

Define $G_{m,n}$ as the collection of pairs of elements in $E^s_{m,n}$:

$$G_{m,n} := \left\{(p,q) \in E^s_{m,n} \times E^s_{m,n} \;\middle|\; r(p_1) = r(q_1),\ r(p_2) = r(q_2)\right\}.$$

Then $G_{m,n}$ is a finite equivalence relation and hence a groupoid.

Proposition 3.26 ([PT22, Corollary 8.10])**.** *There is a short exact sequence*

$$\bigoplus_{v \in V_n} A^+_{m,n,v} \otimes \mathrm{C}_0(0, \nu_s(v)) \rightarrowtail B^+_{m,n} \twoheadrightarrow \mathrm{C}^*(G_{m,n}). \tag{3.27}$$

3.3.3 Comparing $A^{Y+}_{\mathcal{B}}$ with $\mathrm{C}^*(T^+(S^r_{\mathcal{B}}))$, $B^+_{\mathcal{B}}$ with $\mathrm{C}^*(T^\sharp(S_{\mathcal{B}}))$

Now we describe the two isomorphisms given in the diagram (3.13).

Theorem 3.28 ([PT22, Theorem 8.11])**.** *We have isomorphisms $A^{Y+}_{\mathcal{B}} \simeq \mathrm{C}^*(T^+(S^r_{\mathcal{B}}))$ and $B^+_{\mathcal{B}} \simeq \mathrm{C}^*(T^\sharp(S_{\mathcal{B}}))$. More precisely:*

(1) *The map $(\pi^r \times \pi^r)^*\colon \mathrm{C_c}(T^+(S^r_{\mathcal{B}})) \to \mathrm{C_c}(T^+(Y_{\mathcal{B}}))$ induces an isomorphism*

$$A^{Y+}_{\mathcal{B}} \simeq \mathrm{C}^*(T^+(S^r_{\mathcal{B}})).$$

(2) *The map $(\rho^s \times \rho^s)^*\colon \mathrm{C_c}(T^\sharp(S_{\mathcal{B}})) \to \mathrm{C_c}(T^\sharp(S^r_{\mathcal{B}}))$ induces an isomorphism*

$$B^+_{\mathcal{B}} \simeq \mathrm{C}^*(T^\sharp(S_{\mathcal{B}})).$$

3.3.4 $C^+_{\mathcal{B}}$, the inductive system $(C^+_{-n,n})$

Definition 3.29. The foliation C*-algebra is defined as

$$C^+_{\mathcal{B}} := \mathrm{C}^*(\mathcal{F}^+_{\mathcal{B}}).$$

This is a C*-subalgebra of $\mathrm{C}^*(T^\sharp(S_{\mathcal{B}}))$. We will show that, in a very similar fashion with $B^+_{\mathcal{B}}$: the C*-algebra $C^+_{\mathcal{B}}$ can be realised as an inductive limit of C*-algebras $(C^+_{m,n})$, whose K-theory can be computed by a short exact sequence.

Definition 3.30 ([PT22, Proposition 8.7])**.** Define

$$C^+_{m,n} := AC^{Y+}_{m,n} \cap B^+_{\mathcal{B}}.$$

Then every $C^+_{m,n}$ $(m < n)$ is a C*-subalgebra of $C^+_{\mathcal{B}}$.

Proposition 3.31 ([PT22, Theorem 8.13])**.** *Define $C^+_{m,n}$ as above. Then:*

(1) $C^+_{m,n} \subseteq C^+_{m-1,n+1}$. *So $(C^+_{-n,n})$ is an inductive system.*

(2) $C^+_{\mathcal{B}}$ *is the inductive limit* C*-*algebra given by the inductive system, that is,*

$$C^+_{\mathcal{B}} = \overline{\bigcup_n C^+_{-n,n}}.$$

K-theory of $C^+_{m,n}$ can be computed by a short exact sequence, which uses another inductive system of groupoids $(H_{m,n})$, each $H_{m,n}$ being a subgroupoid of $G_{m,n}$. The construction requires the standing assumption. More details can be found in [PT22, Proposition 7.15, Page 57]. The definition there is neither well-organised nor clear enough, so I do not intend to provide a complete description of the groupoids $(H_{m,n})$ here.

Proposition 3.32 ([PT22, Corollary 8.14])**.** *There is a short exact sequence*

$$\bigoplus_{v \in V_n} A^+_{m,n,v} \otimes \mathrm{C}_0(0, \nu_s(v)) \rightarrowtail C^+_{m,n} \twoheadrightarrow \mathrm{C}^*(H_{m,n}). \tag{3.33}$$

4 Fredholm modules and K-theory of Bratteli diagrams

We will study some Fredholm modules and the K-theory of those groupoid C*-algebras introduced in Malte's previous talk, following [PT22, §9–10].

Standing assumptions. Let $\mathcal{B} = (V, E, r, s)$ be a bi-infinite Bratteli diagram, with orders $\leq_s$ and $\leq_r$. Throughout this lecture, we will assume that:

- $\mathcal{B}$ has *finite rank*, that is, V_n is uniformly bounded.
- $\mathcal{B}$ is *strongly simple*, that is, $\mathcal{B}$ is simple and $\#X_v^{\pm} = \infty$ for every $v \in V$. (See Definition 2.6).
- $X_{\mathcal{B}}^{\mathrm{ext}} \cap \partial_r X_{\mathcal{B}} = \emptyset$ and $X_{\mathcal{B}}^{\mathrm{ext}} \cap \partial_s X_{\mathcal{B}} = \emptyset$.

4.1 Fredholm modules

4.1.1 Connes' quantised calculus on fractal spaces

Fredholm modules are among the fundamental objects of noncommutative geometry, which are representatives of K-homology classes. But so far it is not clear to me whether or not their representing K-homology classes play a role in the due paper [PT22]. A quick introduction to Fredholm modules can be found in [NCG22b] and the references therein. For the sake of completeness, we recall the definition here.

Definition 4.1. Let A be a C*-algebra. An even Fredholm module over A is a triple $(\mathcal{H}, \rho, F)$, where

- $\mathcal{H}$ is a separable $\mathbb{Z}/2$-graded Hilbert space[6].
- $\rho\colon A \to \mathbb{B}(\mathcal{H})$ is an even *-homomorphism[7].
- $F \in \mathbb{B}(\mathcal{H})$ is an odd bounded operator[8] satisfying

$$[F, \rho(a)],\ \rho(a)(F^2 - 1),\ \rho(a)(F - F^*) \in \mathbb{K}(\mathcal{H}), \qquad \text{for all } a \in A.$$

An odd Fredholm module over A is defined in a similar way but with all the grading conditions dropped.

Those Fredholm modules defined in [PT22, §9] are motivated by Connes's quantised calculus on fractal spaces [Con94]. Note that if $(A, \mathcal{H}, F)$ is a Fredholm module, then

$$[F, -]\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H}), \qquad T \mapsto [F, T]$$

is a bounded *-derivation. This is the quantised derivation defined by Connes, which ought to be a possible replacement of derivations (differential calculi) on noncommutative manifolds. We will, however, not touch this point here.

Then we are motivated to construct a Fredholm module over the Cantor set and wish that it gives the desired calculus thereof. Let X be the Cantor ternary set, constructed from the Cantor binary tree on $[0, 1]$. This means that we iteratively delete the middle open interval from a set of closed line segments. The process takes infinite but countably many steps, so we may order all those deleted line intervals $(x_n, y_n) \subseteq [0, 1] \setminus X$ in a certain way. This allows us to find a countable dense subspace of X. Namely, the collections

$$\mathcal{X} := \mathcal{X}_+ \sqcup \mathcal{X}_-, \qquad \mathcal{X}_+ := \{x_n\}_{n\in\mathbb{N}}, \qquad \mathcal{X}_- := \{y_n\}_{n\in\mathbb{N}}.$$

[6] That means, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a direct sum of two separable Hilbert spaces.

[7] That means, $\rho(a)$ is a diagonal operator in the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ for every $a \in A$.

[8] Being odd means that F is off-diagonal in the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Define the $\mathbb{Z}/2$-graded Hilbert space

$$\mathcal{H} := \ell^2(\mathcal{X}_+) \oplus \ell^2(\mathcal{X}_-)$$

from this countable subspace but with the discrete measure, that is, the infinite measure on X with point mass at each x_n and y_n. This is the closed linear span of those functions δ_{x_n} and δ_{y_n} supported on a single point.

Let $A := \mathrm{C}(X)$ and $\rho\colon A \to \mathbb{B}(\mathcal{H})$ be the *-representation

$$\rho(f)\delta_{x_n} := f(x_n)\delta_{x_n}, \qquad \rho(f)\delta_{y_n} := f(y_n)\delta_{y_n}.$$

Let $F \in \mathbb{B}(\mathcal{H})$ be defined on the basis by

$$F(\delta_{x_n}) := \delta_{y_n}, \qquad F(\delta_{y_n}) := \delta_{x_n}.$$

Proposition 4.2. *The $(\mathcal{H}, \rho, F)$ defined above is an even Fredholm module over $A = \mathrm{C}(X)$.*

Proof. The only non-trivial part to show is that $[F, f]$ is compact for every $f \in \mathrm{C}(X)$. But this is because every $f \in \mathrm{C}(X)$ can be approximated by locally constant functions due to the compactness of X, and for such functions $[F, f]$ has finite rank. As $\mathbb{K}(\mathcal{H})$ is the norm closure of all finite rank operators, this proves our claim. □

4.1.2 Fredholm modules of a bi-infinite Bratteli diagram

Under the standing assuption, we have the following:

Lemma 4.3 ([PT22, Proposition 7.4])**.** *There exist $I^+_{\mathcal{B}}, J^+_{\mathcal{B}} \in \mathbb{N}$, and infinite paths*

$$x_1, x_2, \ldots, x_{I^+_{\mathcal{B}}}, x_{I^+_{\mathcal{B}}+1}, \ldots, x_{I^+_{\mathcal{B}}+J^+_{\mathcal{B}}} \in Y_{\mathcal{B}},$$

such that:

- *For $1 \le i \le I^+_{\mathcal{B}}$, all but finite many edges in x_i are s-maximal.*
- *For $I^+_{\mathcal{B}} + 1 \le i \le I^+_{\mathcal{B}} + J^+_{\mathcal{B}}$, all but finite many edges in x_i are s-minimal.*
- *There is a decomposition*

$$\partial_s X_{\mathcal{B}} = \bigcup_{i=1}^{I^+_{\mathcal{B}}+J^+_{\mathcal{B}}} T^+(x_i).$$

- *The map Δ_s restricts to a measure-preserving bijection*

$$\Delta_s\colon \bigcup_{1\le i\le I^+_{\mathcal{B}}} T^+(x_i) \xrightarrow{\sim} \bigcup_{I^+_{\mathcal{B}}+1\le j\le I^+_{\mathcal{B}}+J^+_{\mathcal{B}}} T^+(x_j).$$

Definition 4.4 ([PT22, Definition 7.5])**.** Let $I^+_{\mathcal{B}}$ and $J^+_{\mathcal{B}}$ be given as above. Define

$$I^+_{\mathcal{B}} *_\Delta J^+_{\mathcal{B}} := \left\{ (x_j, y_j) \,\middle|\, 1 \le i \le I^+_{\mathcal{B}} < j \le I^+_{\mathcal{B}} + J^+_{\mathcal{B}},\ \Delta_s(T^+(x_i)) \cap T^+(x_j) \neq \emptyset \right\}.$$

Now we are able to define the Fredholm module associated to $\mathcal{B}$, see

Definition 4.5 ([PT22, Definition 9.1])**.** Let $I^+_{\mathcal{B}}$ and $J^+_{\mathcal{B}}$ be defined as above.

- Define $\mathcal{H}_i := L^2(T^+(x_i), \nu_r^{x_i})$, and the $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{B}}^{\min} \oplus \mathcal{H}_{\mathcal{B}}^{\max}$, where

$$\mathcal{H}_{\mathcal{B}}^{\min} := \bigoplus_{1 \le i \le I_{\mathcal{B}}^+} \mathcal{H}_i, \qquad \mathcal{H}_{\mathcal{B}}^{\max} := \bigoplus_{I_{\mathcal{B}}^+ + 1 \le j \le I_{\mathcal{B}}^+ + J_{\mathcal{B}}^+} \mathcal{H}_j.$$

- Define the representation

$$\lambda^x : A_{\mathcal{B}}^+ \to \mathbb{B}(\mathcal{H}_i), \qquad (\lambda^x(f)\xi)(y) := \int_{T^+)x} f(y,z)\xi(z)\mathrm{d}\nu_r^x(z)$$

and the representation

$$\pi : A_{\mathcal{B}}^+ \to \mathbb{B}(\mathcal{H}), \qquad \pi := \bigoplus_{i=1}^{I_{\mathcal{B}}^+ + J_{\mathcal{B}}^+} \lambda^{x_i}.$$

- Define $F : \mathcal{H} \to \mathcal{H}$ as

$$F(\xi)(x) := \xi(\Delta_s(x)).$$

Then F is an self-adjoint, involutive odd operator due to 4 of the previous lemma.

Note that we have inclusions of C*-algebras

$$B_{\mathcal{B}}^+ \subseteq A_{\mathcal{B}}^{Y+} \subseteq A_{\mathcal{B}}^+.$$

The passage from $T^+(x)$ to $T^+(x) \cap (Y_{\mathcal{B}} \times Y_{\mathcal{B}})$ removes a countable closed subset, which has measure zero since ν_r^x is atomless. Then the representations λ^x and π of $A_{\mathcal{B}}^+$ on the corresponding Hilbert spaces give representations of $A_{\mathcal{B}}^{Y+}$ and $B_{\mathcal{B}}^+$ as well.

We want to show that $(\mathcal{H}, \pi, F)$ is an even Fredholm module for either $A = A_{\mathcal{B}}^+, A_{\mathcal{B}}^{Y+}$ or $B_{\mathcal{B}}^+$. Since F is odd, self-adjoint and involutive, it suffices to show that $[F, a] \in \mathbb{K}(\mathcal{H})$ for all $a \in A$. To this end, for each $p \in E_{m,n}$, we are able to define a pair of vectors (see [PT22, Page 47])

$$\xi_p^{\max} \in \mathcal{H}_{\mathcal{B}}^{\max}, \quad \xi_p^{\min} \in \mathcal{H}_{\mathcal{B}}^{\min},$$

such that

$$\mathcal{H}_{\mathcal{B}}^{\max} = \overline{\mathrm{span}}\{\xi_p^{\max} \mid p \in E_{m,n}\}, \quad \mathcal{H}_{\mathcal{B}}^{\min} = \overline{\mathrm{span}}\{\xi_p^{\min} \mid p \in E_{m,n}\}.$$

Proposition 4.6 ([PT22, Proposition 9.3]). *Let $(\mathcal{H}, \pi, F)$ be defined as above. $p, q \in E_{m,n}$ satisfy $r(p) = r(q) = v$. Then:*

(1) *We have*

$$[\pi(a_{p,q}), F] = [|\xi_p^{\max}\rangle\langle\xi_q^{\max}|, F] + [|\xi_p^{\min}\rangle\langle\xi_q^{\min}|, F].$$

So $[\pi(a_{p,q}), F]$ has at most rank 2.[9]

(2) *Let $p, q \in E_{m,n}^Y$ satisfy $r(p) = r(q) = v$ and define the map $a_{p,q} : T^+(Y_{\mathcal{B}}) \to \mathbb{R}$ as in Definition and Lemma 3.19. Recall that $a_{p,q} \in A_{m,n,v}^{Y+}$ (Definition 3.17, Definition and Lemma 3.19). Define $a_{\tilde{p,q}} \in AC_{m,n}^{Y+} := A_{m,n}^{Y+} \otimes C(X_v^+)$ by*

$$\tilde{a}_{p,q} := [x \mapsto \nu_r(v)^{-1}\varphi_s^v(x)] \otimes a_{p,q}, \quad \text{for } x \in X_v^+.$$

Then

$$[\pi(\tilde{a}_{p,q}), F] = [|\xi_p^{\max}\rangle\langle\xi_q^{\max}|, F].$$

[9] Let $\mathcal{H}$ be a Hilbert space and $x, y \in \mathcal{H}$. We write $|x\rangle\langle y|$ for the rank-1 operator $\mathcal{H} \to \mathcal{H}$ sending z to $x\langle y, z\rangle$.

(3) *More generally, given any* $f \in \mathrm{C}([0, \nu_r(v)])$ *and set*

$$\tilde{a}_{p,q} := [x \mapsto f(\varphi_s^v(x))] \otimes a_{p,q}, \quad \text{for } x \in X_v^+.$$

Then

$$\Big[\pi(\tilde{a}_{p,q}), F\Big] = f(0)\Big[|\xi_p^{\min}\rangle\langle\xi_q^{\min}|, F\Big] + f(\nu_r(v))\Big[|\xi_p^{\max}\rangle\langle\xi_q^{\max}|, F\Big].$$

Corollary 4.7 ([PT22, Corollary 9.4, Theorem 9.6])**.** *$(\mathcal{H}, \pi, F)$ is an even Fredholm module over either $A_{\mathcal{B}}^+$, $A_{\mathcal{B}}^{Y+}$ or $B_{\mathcal{B}}^+$. In particular, $a \in A_{\mathcal{B}}^{Y+}$ belongs to $B_{\mathcal{B}}^+$ iff $[F, \pi(a)] = 0$.*

4.2 K-theory of Bratteli diagrams

4.2.1 K-theory of $A_{\mathcal{B}}^+$ and $A_{\mathcal{B}}^{Y+}$

By the AF-structure of $A_{\mathcal{B}}^+$ (Proposition 3.18) and the standard K-theory computation for AF-algebras, the K-theory of $A_{\mathcal{B}}^+$ can be computed from the inductive system $(A_{m,n}^+)$. Each $A_{m,n}^+$ is a finite matrix which can be described by an adjacent matrix given by the bi-infinite Bratteli diagram $\mathcal{B} = (V, E, r, s)$ as follows (see [PT22, Proposition 8.3]).

Let $\mathcal{E}_n$ be the adjacent matrix for $E_n \subseteq E$, that is, the $(\#V_n \times \#V_{n-1})$-matrix with entries

$$(\mathcal{E}_n)_{v,w} := (\text{\# of edges } v \to w,\ v \in V_{n-1},\ w \in V_n).$$

There is an inductive system

$$\mathbb{Z}^{\#V_0} \xrightarrow{\mathcal{E}_1} \mathbb{Z}^{\#V_1} \xrightarrow{\mathcal{E}_2} \cdots.$$

Theorem 4.8 ([PT22, Theorem 10.2])**.** *We have*

$$\mathrm{K}_0(A_{\mathcal{B}}^+) = \lim_{n\to\infty}\left(\mathbb{Z}^{\#V_0} \xrightarrow{\mathcal{E}_1} \mathbb{Z}^{\#V_1} \xrightarrow{\mathcal{E}_2} \cdots\right), \qquad \mathrm{K}_1(A_{\mathcal{B}}^+) = 0.$$

We have claimed that the inclusion $A_{\mathcal{B}}^{Y+} \hookrightarrow A_{\mathcal{B}}^+$ induces an isomorphism in K-theory because they are stably isomorphic. We can say more: the inclusion is indeed an order isomorphism between the ordered K_0-groups $(\mathrm{K}_0(A_{\mathcal{B}}^+), \mathrm{K}_0(A_{\mathcal{B}}^+)_+)$ following the standard result of AF-algebras.

4.2.2 K-theory of $B_{\mathcal{B}}^+$

The K-theory of $B_{\mathcal{B}}^+$ can be computed using the long exact sequence in K-theory given by the extension (3.27).

Theorem 4.9. *We have $\mathrm{K}_1(B_{\mathcal{B}}^+) \simeq \mathbb{Z}$. A generator is given as follows.*

Let $p \in E_{m,n}^Y$ and set $v = r(p) \in V_n$. Choose any continuous function

$$f \colon [0, \nu_r(v)] \to [0,1], \qquad \text{satisfying} \quad f(0) = 1,\ f(\nu_r(v)) = 1.$$

Then

$$u = \exp(2\pi i f \circ \varphi_r^v(x)) a_{p,p} + (1 - a_{p,p})$$

is a generator for $\mathrm{K}_1(B_{\mathcal{B}}^+) \simeq \mathbb{Z}$, where φ_r^v is as in Definition 3.7.

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