Localisation algebras and K-homology

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The following serves as the handout for a seminar talk in the local NCG seminar at Radboud University Nijmegen. I will describe Yu's (original) localisation algebras $\mathrm{C}^*_{\mathrm{L}}(X)$, and explain how they work as a model for K-homology. This is essentially due to the functoriality of $X \mapsto \mathrm{K}_*(\mathrm{C}^*_{\mathrm{L}}(X))$ and the Mayer–Vietoris sequence. A short introduction to the coarse Baum–Connes conjecture is given in the appendix.

The main references for the talk are [Yu97, QR10] and [WY20, Chapter 6–8]. The appendix is based on [HR95].

1. Motivation

Topological K-theory is a generalised cohomology theory for spaces. It follows from abstract nonsense that a dual homology theory exists (Simon's talk, Apr 5). In practice, there are several models for K-homology. A model for K-homology is Fredholm modules (Yufan's talk, Mar 15) and we have seen how elliptic operators give rise to K-homology classes (Peter's talk, Mar 29).

Yu's localisation algebras ([Yu97]) are another interesting model. Why do we bother to care about another model? Roughly speaking, localisation algebras depict the locality of differential operators, making the assembly map more explicit.

Let X be a space. The *assembly map*, or the *higher index map*, is a group homomorphism

$$\mu\colon \mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$$

from the K-homology of X to the K-theory of the Roe C^*-algebra of X.

Traditionally, the map μ is defined using *Paschke duality*. This makes use of another larger C^*-algebra $\mathrm{D}^*(X)$ that contains $\mathrm{C}^*(X)$ as an ideal. The long exact sequence in K-theory gives a connecting homomorphism

$$\partial\colon \mathrm{K}_{*+1}(\mathrm{D}^*(X)/\mathrm{C}^*(X)) \to \mathrm{K}_*(\mathrm{C}^*(X));$$

and Paschke duality (c.f. [Pas81, HR95], Appendix A) implies that

$$\mathrm{K}_*(X) \simeq \mathrm{K}_{*+1}(\mathrm{D}^*(X)/\mathrm{C}^*(X)).$$

The assembly map μ is defined as the composition of the two maps described above.

When X is a "very nice" space, e.g. X is uniformly contractible, has bounded geometry, and coarsely embeds into a Hilbert space, then the assembly map μ is an isomorphism. Convex open subsets in $\mathbb{R}^n$ satisfy such requirements. For example, let $X = \mathbb{R}$. Then $\mathrm{K}_0(\mathbb{R}) = \mathrm{K}_0(\mathrm{C}^*(\mathbb{R})) = 0$ and $\mathrm{K}_1(\mathbb{R}) = \mathrm{K}_1(\mathrm{C}^*(\mathbb{R})) = \mathbb{Z}$. A generator for both $\mathrm{K}_1(\mathbb{R})$ and $\mathrm{K}_1(\mathrm{C}^*(\mathbb{R}))$ is the class of the operator $[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}]$.

In general, however, μ fails to be an isomorphism: the left-hand side of μ is local and topological, whereas the right-hand side is coarse (large scale) and analytic. So the map can be viewed as "assembling" local data to global data. This explains the name.

Locality is closely related to the propagation speed. Very intuitively, being local means the propagation can be arbitrarily small, or has “asymptotically vanishing propagation”. The following explains how this is related to K-homology:

Proposition 1.1 ([WY20, Proposition 6.1.1])**.** *Let X be a compact metric space and $\mathcal{H}_X$ be an X-module. Let $\{T_t\}_{t\in[1,\infty)}$ be a norm-continuous and uniformly bounded family of operators on $\mathcal{H}_X$. Then the following are equivalent:*

- *T_t asymptotically commutes with $\mathrm{C}(X)$. That is, for all $f \in \mathrm{C}(X)$, $\lim_{t\to\infty} \|[T_t, f]\| = 0$.*
- *There exists a norm-continuous family of bounded operators $\{S_t\}_{t\in[1,\infty)}$ on $\mathcal{H}_X$, such that*

$$\lim_{t\to\infty} \mathrm{prop}(S_t) = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} \|T_t - S_t\| = 0.$$

If $\{P_t\}_{t\in[1,\infty)}$ is a norm-continuous family of compact projections on $\mathcal{H}$ that asymptotically commute with $\mathrm{C}(X)$. Then for any projection $q \in A$, we have

$$(P_t q)^2 - P_t q = P_t[q, P_t]q \to 0.$$

Such a family of operators $\{P_t q\}_{t\in[1,\infty)}$ defines a genuine projection $\chi(P_t q)$, which is compact as it is the norm limit of a sequence of compact operators. This gives a group homomorphism

$$\mathrm{K}_0(\mathrm{C}(X)) \to \mathbb{Z}, \qquad [q] \mapsto [\chi(P_t q)].$$

It suggests that family of operators, which asymptotically commute with elements of $\mathrm{C}(X)$, might serve as a model for $\mathrm{K}_0(X)$.

Yu’s localisation algebras $\mathrm{C}^*_{\mathrm{L}}(X)$ make the above ideas of having “asymptotically vanishing propagation” precise, so as to make the locality of K-homology explicit. The assembly map is interpreted as an evaluation map $\mathrm{ev}_1 \colon \mathrm{C}^*_{\mathrm{L}}(X) \to \mathrm{C}^*(X)$.

Remark 1.2. The cases where μ is an isomorphism are quite rare. In a more general setting, one replaces $\mathrm{K}_*(X)$ by a coarse analogue $\mathrm{KX}_*(X)$ and obtain a group homomorphism

$$\mu_\infty \colon \mathrm{KX}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X)).$$

This is the coarse Baum–Connes assembly map. The coarse Baum–Connes conjecture asserts that μ_∞ is an isomorphism. It is known that this is not true in general, with counterexamples violating either the injectivity or surjectivity. Nevertheless it is still interesting to understand under which conditions μ_∞ *is* an isomorphism.

For a quick introduction to the coarse Baum–Connes conjecture, see the Appendix A or the article [HR95].

2. Yu’s localisation algebras

Throughout the talk, all spaces will be assumed to be locally compact, second countable metric spaces, and all maps between them are assumed to be coarse.

Definition 2.1. Let X be a proper metric space, $\mathcal{H}_X$ be an ample X-module. Yu’s localisation algebra is defined as

$$\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X) := \{f \in \mathrm{C}_{\mathrm{ub}}([1,\infty), \mathrm{C}^*(X, \mathcal{H}_X)) \mid \mathrm{prop}(f_t) \to 0\}.$$

where C_{ub} means *uniformly continuous, bounded* functions.

Remark 2.2. As later notations suggest, $\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)$ does not depend on the choice of the *ample* module. However, for the sake of nice functoriality, it is necessary to work with a stronger form of ampleness, referred to as *very ampleness*. This means $\mathcal{H}_X$ is the sum of infinite many copies of ample modules.

Remark 2.3. While preparing for this talk, I realised that there are quite many slightly different versions of localisation algebras, c.f. [Yu97, DWW18, WY20]. The version I present here is Yu's original one, which is similar to the "localised Roe algebras" defined in [WY20, Chapter 6], yet still not the same.

Nevertheless, all these different localisation algebras have isomorphic K-theory, hence are models of K-homology.

We will construct a *local index map*

$$\mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*_{\mathrm{L}}(X))$$

in the even case and show that it is an isomorphism for all proper metric spaces. The odd case is similar.

Let $(\mathcal{H}, F)$ be an even Fredholm module for X. For each $n \in \mathbb{N}_{\geq 1}$, find a locally finite, compactly supported partition of unity $\{\phi_i^n\}_{i\in I}$ on X subordinate to the open $1/n$-balls of X. Define

$$F_n := \sum_{i\in I} \sqrt{\phi_i^n} F \sqrt{\phi_i^n}$$

where the infinite sum is in the strong limit sense; as well as the linear interpolations

$$F_t := (t-n)F_{n+1} + (n+1-t)F_n, \qquad \text{for } n \leq t < n+1.$$

Lemma 2.4. *For every $t \in [1,\infty)$, $(\mathcal{H}_X, F_t)$ is a Fredholm module, which represents the same* K-*homology class as $(\mathcal{H}_X, F)$.*

Proof. It suffices to prove that $F_t - F$ is locally compact, i.e. $a(F_t - F) \in \mathbb{K}(\mathcal{H}_X)$ for all $a \in \mathrm{C}_{\mathrm{c}}(X)$. For every n, notice that

$$\begin{aligned} a(F_n - F) &= a\left(\sum_i \sqrt{\phi_i^n} F \sqrt{\phi_i^n} - F\right) \\ &= a\left(\sum_i \sqrt{\phi_i^n}\left[F, \sqrt{\phi_i^n}\right]\right) \\ &= \sum_i a\sqrt{\phi_i^n}\left[F, \sqrt{\phi_i^n}\right]. \end{aligned}$$

Since $a \in \mathrm{C}_{\mathrm{c}}(X)$, there are finite many i such that $a\sqrt{\phi_i^n} \neq 0$. Therefore, the sum is finite. Since $(\mathcal{H}_X, F)$ is a Fredholm module and $a\sqrt{\phi_i^n} \in \mathrm{C}_{\mathrm{c}}(X)$, we have $a\sqrt{\phi_i^n}\left[F, \sqrt{\phi_i^n}\right]$ is compact for every i. So $a(F_n - F)$ is a finite sum of compact operators and hence compact.

Since F_n is a locally compact perturbation of F, they are operator homotopic through the linear path connecting them. Also $F_t = (t-n)F_{n+1} + (n+1-t)F_n$ is operator homotopic to F. Hence $(\mathcal{H}_X, F_t)$ is a Fredholm module representing the same K-homology class as $(\mathcal{H}_X, F)$. ☐

Proposition 2.5. $(F_t)_{t\in[1,\infty)}$ *is a multiplier of $\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)$ which is invertible modulo $\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)$. Hence it defines a class in $\mathrm{K}_1(\mathcal{M}(\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)/\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)))$ and a class in $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X))$. The class is independent of the choice $\{\phi_i^n\}_{i\in I}$ and the representative of the class of F. Hence $(\mathcal{H}_X, F) \mapsto (F_t)_{t\in[1,\infty)}$ defines a map*

$$\mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)).$$

Proof. Let $f \in \mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)$. We must show that $Ff \in \mathrm{C}^*_{\mathrm{L}}(X, \mathcal{H}_X)$. This means $F_t f_t \in \mathrm{C}^*(X, \mathcal{H}_X)$ for all t and $\mathrm{prop}(F_t f_t) \to 0$. Notice that for every n we have $\mathrm{prop}(F_n) \leq 2/n$: if $f, g \in \mathrm{C}_{\mathrm{c}}(X)$ satisfy $d(\mathrm{supp}(f), \mathrm{supp}(g)) > 2/n$, then $f\sqrt{\phi_i^n}F\sqrt{\phi_i^n}g = 0$ because either $f\sqrt{\phi_i^n} = 0$ or $g\sqrt{\phi_i^n} = 0$. Therefore $\mathrm{prop}(F_t f_t) \leq \mathrm{prop}(F_t) + \mathrm{prop}(f_t) \to 0$ and F_t has finite propagation, too. For local compactness, let $a \in \mathrm{C}_{\mathrm{c}}(X)$. we have $F_t f_t a \in \mathbb{K}$ because $f_t a \in \mathbb{K}$, and

$$aF_t f_t = \sum_i a\sqrt{\phi_i^n} F \sqrt{\phi_i^n} f_t.$$

The sum is finite because a is compactly supported, and each summand is compact because $\sqrt{\phi_i^n} f_t$ is compact. Therefore $F_t f_t$'s are locally compact and have finite propagation. This, together with $\mathrm{prop}(F_t f_t) \to 0$, show that $Ff \in \mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X)$.

Now $a(F_t F_t^* - 1) \in \mathbb{K}$ for all t because $(\mathcal{H}_X, F_t)$ is a Fredholm module. This implies that F_t is unitary modulo $\mathrm{C}_{\mathrm{L}}^*(X)$. Hence it represents a class in $\mathrm{K}_1(\mathcal{M}(\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X)/\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X)))$ and a class in $\mathrm{K}_0(\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X))$ via the boundary map in K-theory long exact sequence. In particular, the boundary map is injective because $\mathrm{K}_1(\mathcal{M}(\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X)) = 0$. Given other choices of partition of unity and construct another F_t, it differs from F_t by an element in $\mathrm{C}_{\mathrm{L}}^*(X)$. Hence the class in $\mathrm{K}_1(\mathcal{M}(\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X)/\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X)))$ and the class in $\mathrm{K}_0(\mathrm{C}_{\mathrm{L}}^*(X, \mathcal{H}_X))$ are indeed well-defined. □

3. Localisation algebras as a model for K-homology

The following result was proven in [Yu97] for all finite-dimensional simplicial complexes, and strengthened to all finite-dimensional proper metric spaces in [QR10]. The proof in [QR10] is elegant but needs Paschke duality. So I plan to talk only about Yu's original proof.

Theorem 3.1 (Yu's theorem)**.** *For any finite-dimensional proper metric space X, the local index map $\mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}_{\mathrm{L}}^*(X))$ is a natural isomorphism.*

This will imply that $\mathrm{K}_*(\mathrm{C}_{\mathrm{L}}^*(X))$ is a model for K-homology and hence $X \mapsto \mathrm{K}_*(\mathrm{C}_{\mathrm{L}}^*(X))$ must be a homology functor too. For a map comparing two such functors, a general strategy works to show that it is an isomorphism:

1. Show that $\mathrm{K}_i(X) \Rightarrow \mathrm{K}_i(\mathrm{C}_{\mathrm{L}}^*(X))$ is a natural transformation.
2. Show that they coincide on a single point.
3. Show that they coincide on any finite simplicial complex using a cut-and-paste technique, namely, the Mayer–Vietoris sequence.
4. Show that they coincide on any compact metric space by showing they preserve inverse limits, and any compact metric space is the inverse limit of a finite simplicial complex.
5. Show that they coincide on locally compact metric spaces by passing to one-point compactification.

We will sketch how to prove 1–3, following [Yu97].

Sketch of the proof of 1

We must first establish the functoriality of $\mathrm{K}_*(\mathrm{C}_{\mathrm{L}}^*(X))$. Recall that Roe C^*-algebras are defined using *ample* modules and the functoriality is implemented by covering isometries. For $\mathrm{C}_{\mathrm{L}}^*(X)$ we need a family of such ingredients.

Definition 3.2. An X-module is *very ample* if it is the direct sum of infinitely many copies of ample modules.

Let $g\colon X \to Y$ be Lipschitz. A uniformly continuous family of isometries $\{V_t\}_{t\in[1,\infty)}$ is said to *cover* g if:

- For all $\phi \in \mathrm{C}_0(Y)$: $\phi V_t - V_t(\phi \circ g) \in \mathbb{K}(\mathcal{H}_X, \mathcal{H}_Y)$.
- $\sup\{d(g(x), y) \mid (x, y) \in \mathrm{supp}(V_t)\} \to 0$ as $t \to \infty$.

Lemma 3.3 (Discussion after [Yu97, Definition 3.3],[QR10, Proposition 3.2])**.** *If $\mathcal{H}_Y$ is a very ample Y-module. Then any Lipschitz map $X \to Y$ admits a uniformly continuous family of isometries $\{V_t\}_{t\in[1,\infty)}$ covering g. Conjugation by V_t gives a *-homomorphism $\mathrm{Ad}_{V_t}\colon \mathrm{C}_{\mathrm{L}}^*(X) \to \mathrm{C}_{\mathrm{L}}^*(Y)$. The induced map $\mathrm{K}_i(\mathrm{C}_{\mathrm{L}}^*(X)) \to \mathrm{K}_i(\mathrm{C}_{\mathrm{L}}^*(Y))$ is independent of the covering isometry.*

Theorem 3.4. *$X \mapsto \mathrm{K}_*(\mathrm{C}_{\mathrm{L}}^*(X))$ is a functor, and there is a natural transformation $\mathrm{K}_i(X) \Rightarrow \mathrm{K}_i(\mathrm{C}_{\mathrm{L}}^*(X))$.*

Proof of 2

Proof. $\mathrm{K}_*(\mathrm{pt})$ is generated by $(\mathcal{H}, F)$ where F is a Fredholm operator with index $+1$. By construction of the local index map, $(\mathcal{H}, F)$ is mapped to the constant family $(F_t := F)_{t\in[1,\infty)}$. It defines a class in $\mathrm{K}_0(\mathrm{C}_{\mathrm{ub}}([1,\infty),\mathbb{K}) \simeq \mathrm{K}_0(\mathbb{K}) \simeq \mathbb{Z}$. The last second map is induced by evalutation at 1, a homotopy invariance argument implies that it is indeed an isomorphism. Now the map sends F to its index, hence an isomorphism. □

Sketch of the proof of 3

We need a Mayer–Vietoris argument for $\mathrm{C}^*_{\mathrm{L}}(X)$ given by "relative localisation algebras".

Definition 3.5. Let $Z \subseteq X$ be a subspace. Define the *relative localisation algebra* $\mathrm{C}^*_{\mathrm{L}}(Z \subseteq X)$ to be the closed subalgebra of $\mathrm{C}^*_{\mathrm{L}}(X)$ generated by $f \in \mathrm{C}^*_{\mathrm{L}}(X)$ such that there exists a positive function

$$c\colon [1,\infty) \to \mathbb{R}_{>0}$$

satisfying

$$\lim_{t\to\infty} c_t = 0 \qquad \text{and} \qquad d((x,y), Z\times Z) \leq c_t \text{ for all } (x,y) \in \mathrm{supp}(f_t).$$

Lemma 3.6 ([Yu97, Lemma 3.10])**.** *The inclusion* $\mathrm{C}^*_{\mathrm{L}}(Z) \hookrightarrow \mathrm{C}^*_{\mathrm{L}}(Z \subseteq X)$ *induces an isomorphim in* K*-theory.*

Lemma 3.7. *Let* X_1 *and* X_2 *be subspaces of* X *with* $X_1 \cup X_2 = X$*. Then* $\mathrm{C}^*_{\mathrm{L}}(X_1 \subseteq X)$ *and* $\mathrm{C}^*_{\mathrm{L}}(X_2 \subseteq X)$ *are ideals of* $\mathrm{C}^*_{\mathrm{L}}(X)$ *and satisfy*

$$\begin{aligned} \mathrm{C}^*_{\mathrm{L}}(X_1 \subseteq X) + \mathrm{C}^*_{\mathrm{L}}(X_2 \subseteq X) &= \mathrm{C}^*_{\mathrm{L}}(X) \\ \mathrm{C}^*_{\mathrm{L}}(X_1 \subseteq X) \cap \mathrm{C}^*_{\mathrm{L}}(X_2 \subseteq X) &= \mathrm{C}^*_{\mathrm{L}}(X_1 \cap X_2 \subseteq X). \end{aligned}$$

Corollary 3.8 (Mayer–Vietoris sequence, [Yu97, Proposition 3.11])**.** *Let* X_1*,* X_2 *be subspaces of* X *satisfying* $X_1 \cup X_2 = X$*. Then there is a cyclic exact sequence*

$$\begin{tikzcd} \mathrm{K}_0(\mathrm{C}^*_{\mathrm{L}}(X_1\cap X_2)) \arrow[r] & \mathrm{K}_0(\mathrm{C}^*_{\mathrm{L}}(X_1)) \oplus \mathrm{K}_0(\mathrm{C}^*_{\mathrm{L}}(X_2)) \arrow[r] & \mathrm{K}_0(\mathrm{C}^*_{\mathrm{L}}(X)) \arrow[d] \\ \mathrm{K}_1(\mathrm{C}^*_{\mathrm{L}}(X)) \arrow[u] & \mathrm{K}_1(\mathrm{C}^*_{\mathrm{L}}(X_1)) \oplus \mathrm{K}_1(\mathrm{C}^*_{\mathrm{L}}(X_2)) \arrow[l] & \mathrm{K}_1(\mathrm{C}^*_{\mathrm{L}}(X_1\cap X_2)) \arrow[l] \end{tikzcd}$$

Proof. K-theory sends pullback diagrams to long exact sequences. Given two ideals in a C^*-algebra $I, J \subseteq A$, there is a long exact sequence

$$\cdots \to \mathrm{K}_i(I\cap J) \to \mathrm{K}_i(I) \oplus \mathrm{K}_i(J) \to \mathrm{K}_i(A) \to \mathrm{K}_{i-1}(I\cap J) \to \cdots .$$

Now let $I = \mathrm{C}^*_{\mathrm{L}}(X_1 \subseteq X)$ and $J = \mathrm{C}^*_{\mathrm{L}}(X_2 \subseteq X)$. The proof is done by identifying $\mathrm{K}_i(\mathrm{C}^*_{\mathrm{L}}(X_i \subseteq X)$ with $\mathrm{K}_i(\mathrm{C}^*_{\mathrm{L}}(X_i))$ and identifying $\mathrm{K}_i(\mathrm{C}^*_{\mathrm{L}}(X_1 \cap X_2 \subseteq X))$ with $\mathrm{K}_i(\mathrm{C}^*_{\mathrm{L}}(X_1\cap X_2))$ using the previous lemma. □

Theorem 3.9 ([Yu97, Theorem 3.2])**.** *Let* X *be a finite-dimensional simplicial complex endowed with the spherical metric. Then the local index map* $\mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*_{\mathrm{L}}(X))$ *is an isomorphism.*

Proof. We prove by induction. For $n = 0$, this is proved in 2. Assume that $\mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*_{\mathrm{L}}(X))$ is an isomorphism for all $(n-1)$-dimensional simplicial complexes. Now let X be an n-dimensional simplicial complex. For every n-simplex Δ of X, let $c(\Delta)$ be its center. Define

$$\Delta_1 := \{x \in \Delta \mid d(x, c(\Delta)) \leq 1/100\}, \qquad \Delta_2 := \{x \in \Delta \mid d(x, c(\Delta)) \geq 1/100\}.$$

Set

$$X_1 := \bigcup\{\Delta_1 \mid \Delta \text{ is an } n\text{-simplex of } X\}, \qquad X_2 := \bigcup\{\Delta_2 \mid \Delta \text{ is an } n\text{-simplex of } X\}$$

Then X_1 is Lipschitz homotopy equivalent to

$$\{c(\Delta) \mid \Delta \text{ is an } n\text{-simplex of } X\}$$

and X_2 is Lipschitz homotopy equivalent to X^{n-1}, both of which are $(n-1)$-simplexes. Notice that $X_1 \cup X_2 = X^n$ and $X_1 \cap X_2 = \partial\Delta_1$. Apply the Mayer–Vietoris sequence and the five lemma and we are done. ☐

4. Application to Cartan–Hadamard manifolds

Using Yu's localisation algebras, the assembly map $\mu\colon \mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ can be expressed as the composition

$$\mathrm{K}_*(X) \xrightarrow{\sim} \mathrm{K}_*(\mathrm{C}^*_{\mathrm{L}}(X)) \xrightarrow{\mathrm{ev}_1} \mathrm{K}_*(\mathrm{C}^*(X)).$$

The first map is the local index map described before.

As an application, we will prove the following theorem.

Theorem 4.1. *Let M be a simply connected Riemannian manifold with non-positive sectional curvature and has bounded geometry. Then the assembly map $\mu\colon \mathrm{K}_*(M) \to \mathrm{K}_*(\mathrm{C}^*(M))$ is an isomorphism.*

A simply connected complete Riemannian manifold with non-positive sectional curvature is called a Cartan–Hadamard manifold.

Proof. We will only prove for K_1; the case for K_0 is similar with a suspension argument.

We show that μ is surjective. Let $u \in \mathrm{C}^*(M)^+$ be invertible. It represents a class in $\mathrm{K}_1(\mathrm{C}^*(M))$. Define a map F as follow: fix a point $x_0 \in M$. Since M is simply connected and has non-positive sectional curvature, by Cartan–Hadamard Theorem, every two points are connected by a unique geodesic. Define $F_n(x)$ to the unique point on the geodesic connecting x and x_0 satisfying $d(F_n(x), x_0) = d(x, x_0)/2^{n-1}$. The map F_n is Lipschitz because M has non-positve sectional curvature (c.f. [Yu95, Lemma 5.1]).

For every $n \in \mathbb{N}_{\geq 1}$, find an isometry V_n covering F_n (c.f. Malte's talk on Roe C*-algebras). Let

$$V_t := R_{t-n+1} \begin{pmatrix} V_n & \\ & V_{n+1} \end{pmatrix} R^*_{t-n+1}, \qquad \text{if } n \leq t < n+1$$

where

$$R_t := \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Notice that V_t is not a continuous family of isometries. Nevertheless

$$u_t := V_t \begin{pmatrix} u & \\ & 1 \end{pmatrix} V_t^*$$

is a uniformly continuous and bounded family of elements in $\mathrm{C}^*(M)^+$. Since V_n covers F_n, we have $d(y, F_n(x)) < 1/n$ for all $(y, x) \in \mathrm{supp}(V_n)$. Therefore, $\mathrm{prop}(u_t) \to 0$. Clearly u_t is invertible in $\mathrm{C}^*_{\mathrm{L}}(M)$ for every t. Thus $u_t \in \mathrm{C}^*_{\mathrm{L}}(M)^+$ is an invertible element lifting u. Hence $(\mathrm{ev}_1)_*[u_t] = [u]$.

To show μ is injective, let $u_t \in \mathrm{C}^*_{\mathrm{L}}(M)^+$ be an invertible element such that $(\mathrm{ev}_1)_*[u_t] = [1]$. That means there is a path of invertibles $(w_t)_{t\in[0,1]}$ inside $\mathrm{C}^*(M)^+$ with $w_1 = u_1$ and $w_0 = 1$. For each $n \in \mathbb{N}_{\geq 1}$, define $(v_t^{n\prime})_{t\in[1,\infty)}$ as:

$$v_t^{n\prime} := \begin{cases} u_{t-n} & \text{if } t \in [n+1, \infty); \\ w_{t-n} & \text{if } t \in [n, n+1]; \\ 1 & \text{if } t \in [1, n]. \end{cases}$$

Then $v_t^{n\prime}$ is invertible for all t and $\mathrm{prop}(v_t^{n\prime}) \to 0$ as $t \to \infty$. Using the same technique as last paragraph we may find $v_t^n \in \mathbb{M}_2(\mathrm{C}_\mathrm{L}^*(M)^+)$ such that $\mathrm{prop}(v_t^n) < \frac{1}{n+1}$ for every t, and that $\{v_t^n\}_{t\in[1,\infty)}$ represents the same class of $\{v_t^{n\prime}\}_{t\in[1,\infty)}$.

Now we define two elements $(a_t)_{t\in[1,\infty)}, (b_t)_{t\in[1,\infty)} \in \mathrm{C}_\mathrm{L}^*(M, \oplus_\mathbb{N}\mathcal{H}_M)$:

$$a_t := v_t^1 \oplus v_t^2 \oplus v_t^3 \oplus \cdots, \quad b_t := 1 \oplus v_t^2 \oplus v_t^3 \oplus \cdots$$

Then $a_{t-1} = v_t^2 \oplus v_t^3 \oplus \cdots = 1 \oplus b_t$, and a_{t-1} is homotopic to a_t via a path of invertibles $[0,1] \ni \nu \mapsto a_{t-\nu}$. This implies that $[a_t] = [1 \oplus b_t] = [b_t]$ in $\mathrm{K}_1(\mathrm{C}^*(M))$. Notice that

$$ab^{-1} = v_t^1 \oplus 1 \oplus 1 \oplus \cdots,$$

and

$$v_t^1 = \begin{cases} u_{t-1} & \text{if } t \in [2,\infty) \\ w_t & \text{if } t \in [1,2] \end{cases}$$

is homotopic to u_t in $\mathrm{C}_\mathrm{L}^*(X)^+$. This implies that $[u_t] = [ab^{-1}] = [1]$ in $\mathrm{K}_1(\mathrm{C}_\mathrm{L}^*(M))$. □

A. Coarse Baum–Connes conjecture via Paschke duality

The appendix was originally a part of my master thesis, but various changes were made so far. The main reference for this appendix is [HR95].

The coarse Baum–Connes conjecture, proposed by John Roe ([Roe93, Conjecture 6.30]), is an analogue of the celebrated Baum–Connes conjecture in coarse geometry. It states that the coarse Baum–Connes assembly map

$$\mu_\infty \colon \mathrm{KX}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$$

between the coarse K-homology of a proper metric space X and the K-theory of the Roe C*-algebra $\mathrm{C}^*(X)$, is an isomorphism for any "reasonable" space X. The coarse K-homology of X is given by the K-homology of a "universal coarsening" of X.

Roe already realised that the conjecture is unlikely to be true in general, and several counterexamples are later constructed to show that μ_∞ can either be not injective, or be not surjective (c.f. [WY20, Chapter 13]). Nevertheless, enumerous cases when the conjecture holds true are known. In particular, the injectivity of μ_∞ is a form of the Novikov conjecture. This makes the coarse Baum–Connes conjecture still appealing.

In the following, we will construct the coarse Baum–Connes assembly map using Paschke duality.

Definition A.1. Let $\mathcal{H}$ be an ample X-module. An operator $T \in \mathbb{B}(\mathcal{H})$ is *pseudolocal*, if $fTg \in \mathbb{K}(\mathcal{H})$ whenever $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$.

The C*-algebra $\mathrm{D}^*(X)$ is the norm closure of all bounded operators on $\mathcal{H}$ which are pseudolocal and have finite propagation.

Clearly, $\mathrm{D}^*(X)$ contains $\mathrm{C}^*(X)$ as an ideal. Then there is an extension of C*-algebras

$$\mathrm{C}^*(X) \rightarrowtail \mathrm{D}^*(X) \twoheadrightarrow \mathrm{D}^*(X)/\mathrm{C}^*(X),$$

which induces a long exact sequence in K-theory:

$$\cdots \to \mathrm{K}_*(\mathrm{C}^*(X)) \to \mathrm{K}_*(\mathrm{D}^*(X)) \to \mathrm{K}_*(\mathrm{D}^*(X)/\mathrm{C}^*(X)) \xrightarrow{\partial} \mathrm{K}_{*-1}(\mathrm{C}^*(X)) \to \cdots. \tag{1}$$

Theorem A.2 (Paschke duality)**.** *There is an isomorphism* $\mathrm{K}_*(\mathrm{D}^*(X)/\mathrm{C}^*(X)) \simeq \mathrm{K}_{*-1}(X)$. *Composing with the boundary map yields a group homomorphism*

$$\mu_* \colon \mathrm{K}_*(X) \xrightarrow{\sim} \mathrm{K}_{*+1}(\mathrm{D}^*(X)/\mathrm{C}^*(X)) \xrightarrow{\partial} \mathrm{K}_*(\mathrm{C}^*(X))$$

called the assembly map *or the* higher index map.

Proof. Let $\mathcal{H}_X$ be an ample X-module. Paschke proved in [Pas81] that there are isomorphisms

$$\mathrm{K}_i(X) \simeq \mathrm{K}_{1-i}\left(\frac{\text{pseudolocal operators on } \mathcal{H}}{\text{locally compact operators on } \mathcal{H}}\right)$$

for $i = 0, 1$ as follows. For $i = 1$, a class of the right-hand side is given by a pseudolocal projection P. It lifts to a self-adjoint pseudolocal operator Q. Then $(\mathcal{H}, 2Q - 1)$ is an odd Fredholm module for X. For $i = 0$, a class of the right-hand side is given a pseudolocal unitary U which lifts to a pseudolocal operator F. Then $(\mathcal{H}, F)$ is an even Fredholm module for X. The equivalence relations are precisely such that locally compact operators are mapped to zero.

Higson and Roe showed that every pseudolocal operator can be written as the sum of an operator with finite propagation, and a locally compact operator. Therefore, there is an isomorphism

$$\mathrm{D}^*(X)/\mathrm{C}^*(X) \simeq \frac{\text{pseudolocal operators on } \mathcal{H}}{\text{locally compact operators on } \mathcal{H}}$$

as desired. ☐

We cannot expect that μ is an isomorphism in general: the left-hand side depends only on the *local* property of X, whereas the right-hand side is *large scale*. A more general formulation replaces the K-homology by the coarse K-homology. The coarse K-homology of X is defined to be the K-homology of a "universal coarsening" of X. Such a universal coarsening can be taken to be the Rips complex (c.f. [WY20, Chapter 7]) or the direct limit of the anti-Čech system (c.f. [HR95]). We present the latter here.

Definition A.3 (anti-Čech system[1]). Let X be a proper metric space. We say an open cover $\mathcal{U}$ of X is *good* if $\mathcal{U}$ is locally finite and every $U \in \mathcal{U}$ is relatively compact in X. An *anti-Čech system* over X is a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of good covers of X, such that there exists an increasing sequence of real numbers $\{R_n\}_{n\in\mathbb{N}}$ satisfying:

- $R_n \to \infty$.
- Every $U \in \mathcal{U}_n$ has diameter less or equal than R_n.
- The Lebesgue number of $\mathcal{U}_{n+1}$ is greater than R_n. Recall that the *Lebesgue number* of a cover $\mathcal{U}$ of X is the infimum of all numbers $R > 0$ such that: for every subset $V \subseteq X$ whose diameter is smaller than r, there exists $U \in \mathcal{U}$ such that $V \subseteq U$.

The *refinement map* $r_n \colon \mathcal{U}_n \to \mathcal{U}_{n+1}$ sends each $U \in \mathcal{U}_n$ to the open set $U' \in \mathcal{U}_{n+1}$ which contains U as a subset.

Definition A.4 (nerve of a cover). Let $\mathcal{U} = \{U_\alpha\}_\alpha$ be a good cover of X. The *nerve* of $\mathcal{U}$, denoted by $|\mathcal{U}|$, is the simplicial complex defined as follows:

- vertices are labelled by elements $U_\alpha \in \mathcal{U}$.
- n-simplices are $(n+1)$-tuples $(U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n})$ such that $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \neq \emptyset$.

Since $\mathcal{U}$ is a good cover, $|\mathcal{U}|$ is a finite simplicial complex. Equip $|\mathcal{U}|$ with the spherical metric.

Lemma A.5 ([Roe91, Section 3]). *Let X be a complete metric space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be a good cover of X such that:*

- *The Lebesgue number of $\mathcal{U}$ is greater than zero.*

[1] The name comes from the following fact: the Čech complex is formed by refining open covers and the Čech cohomology is obtained by taking the inverse limit, whereas in an anti-Čech system one coarsens open covers and work with the direct limit.

- *Every $U_\alpha \in \mathcal{U}$ is bounded.*

Let $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ be a partition of unity subordinate to $\mathcal{U}$. Then the map

$$\kappa\colon X \to |\mathcal{U}|, \qquad \kappa(x) = \sum_{\alpha\in\Lambda} \varphi_\alpha(x) U_\alpha,$$

is a coarse equivalence.

Corollary A.6. *Let X be a complete metric space and let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be an anti-Čech system over X. Then X is coarsely equivalent to $|\mathcal{U}_n|$ for every n. In particular, $\mathrm{C}^*(X) \simeq \mathrm{C}^*(|\mathcal{U}_n|)$.*

Notice that the refinement $r_n\colon \mathcal{U}_n \to \mathcal{U}_{n+1}$ induces a map $|\mathcal{U}_n| \to |\mathcal{U}_{n+1}|$ between their nerves. For each $|\mathcal{U}_n|$, we construct the assembly map

$$\mu_n\colon\ \mathrm{K}_*(|\mathcal{U}_n|) \to \mathrm{K}_*(\mathrm{C}^*(|\mathcal{U}_n|)) \xrightarrow{\sim} \mathrm{K}_*(\mathrm{C}^*(X))$$

In particular, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}_*(X) \arrow[r, "\kappa_*"] \arrow[drrr, "\mu"'] & \mathrm{K}_*(|\mathcal{U}_1|) \arrow[r, "r_{1*}"] \arrow[drr, "\mu_1"] & \mathrm{K}_*(|\mathcal{U}_2|) \arrow[r, "r_{2*}"] \arrow[dr, "\mu_2"] & \cdots \arrow[r, "r_{n-1*}"] & \mathrm{K}_*(|\mathcal{U}_n|) \arrow[r, "r_{n*}"] \arrow[dl, "\mu_n"'] & \cdots \\
 & & & \mathrm{K}_*(\mathrm{C}^*(X)) & &
\end{tikzcd}$$

where κ_* is the map induced by $\kappa\colon X \to |\mathcal{U}_1|$ in Lemma A.5.

Definition A.7. Let X be a proper metric space and let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be an anti-Čech system over X. The *coarse* K*-homology* of X is

$$\mathrm{KX}_*(X) := \varinjlim \mathrm{K}_*(|\mathcal{U}_n|).$$

The *coarse Baum–Connes assembly map* is the universal map

$$\mu_\infty\colon\ \mathrm{KX}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X)),$$

which fits in the diagram above.

Coarse Baum–Connes conjecture. *For a proper metric space X: the coarse Baum–Connes assembly map μ_∞ is an isomorphism.*

In the end, we comment on some cases when the assembly map $\mu\colon \mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ is an isomorphism. This is true if the following holds:

1. The maps κ_* and r_{n*} are isomorphisms for all n. So $\mathrm{KX}_*(X) \simeq \mathrm{K}_*(X)$.
2. The coarse Baum–Connes assembly map $\mu_\infty\colon\ \mathrm{KX}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ is an isomorphism.

Theorem A.8 ([HR95, Proposition 4.3])**.** *Let Y be a finite-dimensional compact metric space. The conditions 1–2 holds for $X = \mathcal{O}Y$, the open cone over Y. Therefore, the assembly map $\mu\colon \mathrm{K}_*(\mathcal{O}Y) \to \mathrm{K}_*(\mathrm{C}^*(\mathcal{O}Y))$ is an isomorphism.*

Another example is proper metric spaces, which are uniformly contractible, have bounded geometry, and coarsely embeds into Hilbert spaces, e.g. convex open sets in $\mathbb{R}^n$.

Definition A.9. Let X be a proper metric space.

- X is *uniformly contractible*, if for all $R \geq 0$, there exists $S \geq R$, such that the inclusion $\mathrm{B}(x,R) \hookrightarrow \mathrm{B}(x,S)$ is nullhomotopic for all $x \in X$.

- X has *bounded geometry*, if X is coarsely equivalent to a proper metric space Y which is *uniformly locally finite*: for any $R \geq 0$, there exists $N \in \mathbb{N}$ such that $\mathrm{B}(y, R)$ has cardinality at most N for all $y \in Y$.

- Let $\mathcal{H}$ be a separable Hilbert space. A map $\iota\colon X \to \mathcal{H}$ is a *coarse embedding* if there exists non-decreasing functions $\rho_1, \rho_2\colon [0, \infty) \to \mathbb{R}$ such that

$$\rho_1(d(x, y)) \leq \|\iota(x) - \iota(y)\| \leq \rho_2(d(x, y)) \quad \text{for all } x, y \in X,$$

and

$$\lim_{r \to \infty} \rho_i(r) = +\infty \quad \text{for } i = 1, 2.$$

Uniform contractibility is stronger than contractiblity. Any uniformly contractible CW-complex is contractible by induction over skeleta. The subspace in $\mathbb{R}^2$ as shown in Figure 1 is contractible, but not uniformly contractible since some open R-neighbourhoods might even be disconnected.

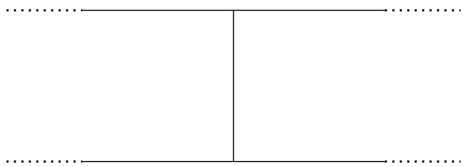

Figure 1: A contractible but not uniformly contractible space

Theorem A.10 ([HR95, Proposition 3.8],[Yu00])**.** *Let X be a proper metric space with bounded geometry.*

- *If X is uniformly contractible. Then the coarsening map $\mathrm{K}_*(X) \to \mathrm{KX}_*(X)$ is an isomorphism. Therefore, for such spaces the coarse Baum–Connes assembly map μ_∞ coincides with the assembly map μ.*

- *If X coarsely embeds into a separable Hilbert space. Then the coarse Baum–Connes assembly map $\mu_\infty\colon \mathrm{KX}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ is an isomorphism.*

Corollary A.11. *If X is a convex set in $\mathbb{R}^n$, then the assembly map $\mu\colon \mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ is an isomorphism.*

Proof. $\mathbb{R}^n$ has bounded geometry since it is coarsely equivalent to $\mathbb{Z}^n$, which is uniformly locally finite. $\mathbb{R}^n$ isometrically embeds into, and hence coarsely embeds into, any separable Hilbert space. Any convex set in $\mathbb{R}^n$ is uniformly contractible. $\square$

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