Completely bounded maps

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This handout is provided for a seminar talk on operator spaces at Leiden University in 2021 winter semester. In this talk I will cover Chapter 8 of Paulsen's book "Completely bounded maps and operator algebras" [Pau02]. I will extend the results of completely positive maps in previous chapters to completely bounded maps using Paulsen's off-diagonal trick, and study bimodule maps, where completely bounded maps are just bounded maps.

Notations and Conventions

- In this handout, $A, B, C, \ldots$ will denote C^*-algebras, M will denote an operator space, and S will denote an operator system. All C^*-algebras are assumed to be **unital**. Calligraphic letters $\mathcal{H}, \mathcal{K}$ will be used for Hilbert spaces.
- For simplicity, we will write cb for completely bounded, cp for complete positive.

Contents

1 Extension and representation theorems for cb maps

In previous chapters, we have seen two important theorems for cp maps: Averson's extension theorem and Stinespring's representation theorem (see Table 1). Paulsen [Pau84] used an "off-diagonal embedding" to identify a cb map with an off-diagonal entry of a cp map. This allows us to extend both theorems to cb maps.

1.1 Wittstock's extension theorem

Let $M \subseteq A$ be an operator space. Define the operator system

$$S_M := \left\{ \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{C},\, a, b \in M \right\}.$$

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Cp maps	Cb maps
Averson's extension theorem Let $S \subseteq A$ be an operator system. Any cp map $S \to \mathbb{B}(\mathcal{H})$ extends to A.	Wittstock's extension theorem Let $M \subseteq A$ be an operator space. Any cb map $M \to \mathbb{B}(\mathcal{H})$ extends to A norm-preservingly.
Stinespring's representation theorem Let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be cp. There exists a representation $\pi\colon A \to \mathbb{B}(\mathcal{K})$ on some Hilbert space $\mathcal{K}$, and a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$, such that $\phi(a) = V^*\pi(a)V$ for all a.	Generalised Stinespring's representation theorem Let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be cb. There exists a representation $\pi\colon A \to \mathbb{B}(\mathcal{K})$ on some Hilbert space $\mathcal{K}$, and bounded operators $V_1, V_2\colon \mathcal{H} \to \mathcal{K}$, such that $\phi(a) = V_1^*\pi(a)V_2$ for all a, and $\|\phi\|_{\mathrm{cb}} = \|V_1\|\|V_2\|$.

Table 1: Comparison: extension and representation theorems of cp and cb maps.

The following lemma identifies any completely contractive map (and even further, cb maps) with an off-diagonal entry of a cp map (written as a 2×2-matrix):

Lemma 8.1

Let $M \subseteq A$ be an operator space. Then $\phi\colon M \to B$ is completely contractive iff $\hat{\phi}\colon S_M \to \mathbb{M}_2(B)$ is cp, where

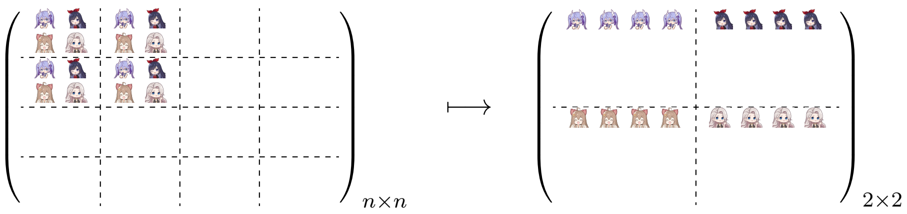
$$\hat{\phi}\begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi(b)^* & \mu \end{pmatrix}.$$

For the proof we need a shuffling technique. Let $\phi\colon A \to B$ be linear, then $\phi_n\colon \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ can be identified with the map $\phi \otimes \mathrm{id}\colon A \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n$. Here $A \otimes \mathbb{M}_n$ denotes the C*-algebraic tensor product of A and $\mathbb{M}_n$, that is, the completion of the algebraic tensor product of the algebras A and $\mathbb{M}_n$ under a suitable C*-norm. There are different C*-norms for tensor algebras in general, yielding different C*-algebraic tensor products. But there is a unique C*-norm on $A \otimes \mathbb{M}_n$ since $\mathbb{M}_n$ is nuclear.

Applying this to the map $\hat{\phi}_n\colon \mathbb{M}_n(S_M) \to \mathbb{M}_n(\mathbb{M}_2(B))$ we identify it with a map which maps from a subspace in $\mathbb{M}_2(\mathbb{M}_n(A))$ to $\mathbb{M}_2(\mathbb{M}_n(B))$ as in the following diagram:

$$\begin{tikzcd}
\mathbb{M}_n(S_M) \arrow[r, "\hat{\phi}_n"] \arrow[d, hook] & \mathbb{M}_n(\mathbb{M}_2(B)) \arrow[ddd, "\cong"] \\
\mathbb{M}_n(\mathbb{M}_2(A)) \arrow[d, "\cong"] & \\
A \otimes \mathbb{M}_2 \otimes \mathbb{M}_n \arrow[d, "\cong"] & \\
\mathbb{M}_2(\mathbb{M}_n(A)) \arrow[r, "\hat{\phi}_n"] & \mathbb{M}_2(\mathbb{M}_n(B))
\end{tikzcd}$$

The identification $\mathbb{M}_n(\mathbb{M}_2(A)) \cong \mathbb{M}_2(\mathbb{M}_n(A))$ is given by a "shuffling of matrix entries" as follows:



The shuffling technique provides a simple way to deal with complete boundedness and positivity.

Proof

Let ϕ be completely contractive. We claim that $\hat{\phi}$ is cp. Consider a positive element in $\mathbb{M}_n(S_M) \subseteq \mathbb{M}_n(\mathbb{M}_2(A)) \cong \mathbb{M}_2(\mathbb{M}_n(A))$. Under the shuffling it is identified with the matrix

$$X := \begin{pmatrix} H & A \\ B^* & K \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n(A)),$$

where H and K are scalar matrices (that is, their entries belong to $\mathbb{C} \subseteq A$). We want to show that

$$\hat{\phi}_n(X) = \begin{pmatrix} H & \phi_n(A) \\ \phi_n(B)^* & K \end{pmatrix}$$

is positive. Since X is positive, it holds that $A = B$, and the matrices H and K are positive. We may assume that H and K are both invertible; otherwise replace them by $H + \epsilon$ and $K + \epsilon$ for some positive real number ϵ.

We have

$$\begin{pmatrix} 1 & H^{-1/2}AK^{-1/2} \\ K^{-1/2}A^*H^{-1/2} & 1 \end{pmatrix} = \begin{pmatrix} H^{-1/2} & \\ & K^{-1/2} \end{pmatrix} \begin{pmatrix} H & A \\ A^* & K \end{pmatrix} \begin{pmatrix} H^{-1/2} & \\ & K^{-1/2} \end{pmatrix}$$

is positive. By Lemma 3.1: $\|H^{-1/2}AK^{-1/2}\| \leq 1$. Now

$$\begin{pmatrix} H & \phi_n(A) \\ \phi_n(A)^* & K \end{pmatrix} = \begin{pmatrix} H^{1/2} & \\ & K^{1/2} \end{pmatrix} \begin{pmatrix} 1 & H^{-1/2}\phi_n(A)K^{-1/2} \\ K^{-1/2}\phi_n(A)^*H^{-1/2} & 1 \end{pmatrix} \begin{pmatrix} H^{1/2} & \\ & K^{1/2} \end{pmatrix}.$$

But $H^{-1/2}\phi_n(A)K^{-1/2} = \phi_n(H^{-1/2}AK^{-1/2})$ (for this: notice that both H and K are **scalar matrices**, and write ϕ_n as $\phi \otimes \mathrm{id} \colon M \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n$.) Then the top-right entry is $\phi_n(H^{-1/2}AK^{-1/2})$. This is contractive since ϕ is completely contractive. By Lemma 3.1, $\hat{\phi}_n(X)$ is positive.

Lemma 8.1 identifies a cb map with the corner of a cp map, and allows a generalisation of Averson's extension theorem.

Theorem 8.2 (Wittstock's extension theorem)

Let $M \subseteq A$ be an operator space, $\phi \colon M \to \mathbb{B}(\mathcal{H})$ be a cb map. There exists a cb map $\psi \colon A \to \mathbb{B}(\mathcal{H})$ extending ϕ with $\|\psi\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$.

Proof

Without loss of generality we assume that $\|\phi\|_{\mathrm{cb}} = 1$. Then by the previous lemma there exists a cp map $\hat{\phi} \colon S_A \to \mathbb{M}_2(\mathbb{B}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. Since $S_A \subseteq \mathbb{M}_2(A)$ is an operator system, by Averson's extension theorem $\hat{\phi}$ extends to $\psi \colon \mathbb{M}_2(A) \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ with $\|\hat{\psi}\| = \|\hat{\phi}\|$.

We define $\psi \colon A \to \mathbb{B}(\mathcal{H})$ to be the top-right entry of $\hat{\psi}$, that is,

$$\hat{\psi}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} * & \psi(a) \\ * & * \end{pmatrix}.$$

The map ψ extends ϕ by construction. It suffices to show that $\|\psi\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}} = 1$. We first show $\|\psi\| = \|\phi\|$ as the completely bounded norm can be worked out in a similar way using the shuffling technique. We have

$$\|\psi(a)\| \leq \left\| \hat{\psi} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \right\| \leq \|\hat{\psi}\| \cdot \|a\| \leq \|a\|.$$

Here we use $\hat{\psi}$ is cp, hence $\|\hat{\psi}\| = \hat{\psi}(1) = 1$. This shows ψ is contractive.

Similarly, we see that ψ is completely contractive:

$$\|\psi_n(A)\| \leq \left\| \hat{\psi}_n \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \right\| \leq \|\hat{\psi}_n\| \cdot \|A\| \leq \|A\|.$$

Therefore ψ extends ϕ and $\|\psi\|_{\text{cb}} = \|\phi\|_{\text{cb}}$.

1.2 Generalised Stinespring's representation theorem

Now we generalise Stinespring's representation theorem to cb maps $\phi\colon A \to \mathbb{B}(\mathcal{H})$. For this we need to understand the extension $\hat{\phi}\colon \mathbb{M}_2(A) \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ in a better sense.

Lemma 8.3

Let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be cb. Then there exist a cp map $\hat{\phi}\colon \mathbb{M}_2(A) \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ of the form

$$\hat{\phi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix},$$

where $\phi_1\colon A \to \mathbb{B}(\mathcal{H})$ and $\phi_2\colon A \to \mathbb{B}(\mathcal{H})$ are cp with $\|\phi_1\|_{\text{cb}} = \|\phi_2\|_{\text{cb}} = \|\phi\|_{\text{cb}}$.

Here $\phi^*\colon A \to \mathbb{B}(\mathcal{H})$ is defined as $\phi^*(a^*) := \phi(a)^*$.

Proof

Again we may assume $\|\phi\|_{\text{cb}} = 1$. Then by Lemma 8.1 we obtain a cp map $\hat{\phi}\colon S_A \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$, and extend it to $\hat{\phi}\colon \mathbb{M}_2(A) \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ by Averson's extension theorem. Now

$$\hat{\phi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \hat{\phi} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \hat{\phi} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \hat{\phi} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}.$$

We claim that

$$\hat{\phi} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \phi_1(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\phi} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \phi_2(d) \end{pmatrix}$$

for some cp maps ϕ_1, ϕ_2 with $\|\phi_1\|_{\text{cb}} = \|\phi_2\|_{\text{cb}} = \|\phi\|_{\text{cb}}$. Take $a \in A$ with $0 < a < 1$. Since $\hat{\phi}$ is cp, we have

$$0 < \hat{\phi} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} < 1.$$

This implies that $\hat{\phi}\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ has only one non-zero entry on the top-left corner. This defines a positive linear map ϕ_1. Since A is spanned by contractive positive elements, ϕ_1 extends to the whole of A. A similar result holds for ϕ_2. All discussion above can be extended to the cp case using a shuffling technique.

The only remaining thing is to show that both ϕ_1 and ϕ_2 have cb-norm 1. But this is because they are cp and $\phi_1(1) = \phi_2(1) = 1$. So $\|\phi_1\|_{\text{cb}} = \|\phi_2\|_{\text{cb}} = 1$ and the proof is finished.

Now we can generalise Stinespring's representation theorem to cb maps:

Theorem 8.4 (Generalised Stinespring's representation theorem)

Let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be cb. Then there exists a Hilbert space $\mathcal{K}$, a representation $\pi\colon A \to \mathbb{B}(\mathcal{K})$ and bounded operators $V_1, V_2\colon \mathcal{H} \to \mathcal{K}$, such that

$$\phi(a) = V_1^* \pi(a) V_2$$

for all $a \in A$ and $\|\phi\|_{\text{cb}} = \|V_1\|\|V_2\|$. If $\|\phi\|_{\text{cb}} = 1$, then V_1 and V_2 can be chosen to be isometries.

Proof

Again we let $\|\phi\|_{\text{cb}} = 1$. Lemma 8.3 provides a cp map

$$\hat{\phi}\colon \mathbb{M}_2(A) \to \mathbb{M}_2(\mathbb{B}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H} \oplus \mathcal{H}), \quad \hat{\phi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix}.$$

Using Stinespring's representation theorem we find a representation $\hat{\pi}\colon \mathbb{M}_2(A) \to \mathbb{B}(\hat{\mathcal{K}})$ and an isometry $V\colon \mathcal{H} \oplus \mathcal{H} \to \hat{\mathcal{K}}$ for some Hilbert space $\hat{\mathcal{K}}$ such that

$$\begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix} = \hat{\phi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = V^* \hat{\pi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} V.$$

A trick allows us to rewrite $\hat{\pi}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as a 2×2 matrix: we decompose

$$\hat{\mathcal{K}} = \operatorname{Ran}\left(\hat{\pi}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \oplus \operatorname{Ran}\left(\hat{\pi}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) =: \mathcal{K} \oplus \mathcal{K}$$

Under this decomposition, $\hat{\pi}\colon \mathbb{M}_2(A) \to \mathbb{B}(\hat{\mathcal{K}}) \cong \mathbb{M}_2(\mathbb{B}(\mathcal{K}))$ can be identified with $\pi \otimes \mathrm{id}\colon A \otimes \mathbb{M}_2 \to \mathbb{B}(\mathcal{K}) \otimes \mathbb{M}_2$ for some π. Thus

$$\begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix} = V^* \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} V. \tag{1}$$

Notice that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = V^* \begin{pmatrix} \pi(1) & 0 \\ 0 & 0 \end{pmatrix} V, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = V^* \begin{pmatrix} 0 & 0 \\ 0 & \pi(1) \end{pmatrix} V.$$

Some straightforward computation implies that V must be diagonal, that is,

$$V = \begin{pmatrix} V_1 & \\ & V_2 \end{pmatrix}$$

for some isometries $V_1, V_2\colon \mathcal{H} \to \mathcal{K}$. Then (1) becomes

$$\begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix} = \begin{pmatrix} V_1^*\pi(a)V_1 & V_1^*\pi(b)V_2 \\ V_2^*\pi(c)V_1 & V_2^*\pi(d)V_2 \end{pmatrix}.$$

Taking the top-right corner yields the result.

Remark

Here we comment that, unlike the Stinespring's representation theorem for cp maps, there is no "uniqueness" statement on the representation (up to unitary equivalence). The reason is that the extension of the map $\hat{\phi}\colon S_A \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ to the whole of $\mathbb{M}_2(A)$ is by no means unique, and the different choices yield different representations.

Stinespring's representation theorem also implies the following (somewhat surprising) result:

Theorem 8.5 (Wittstock's decomposition theorem)

Let $\phi\colon A \to \mathbb{B}(\mathcal{H})$ be cb. Then there exists a cp map $\psi\colon A \to \mathbb{B}(\mathcal{H})$ with $\|\psi\|_{\mathrm{cb}} \le \|\phi\|_{\mathrm{cb}}$, and

$$\psi \pm \operatorname{Re}\phi, \quad \psi \pm \operatorname{Im}\phi$$

are cp. Here

$$\operatorname{Re}\phi := \frac{1}{2}(\phi + \phi^*), \quad \operatorname{Im}(\phi) := \frac{1}{2i}(\phi - \phi^*).$$

In particular, this implies that cb maps from a C*-algebra into $\mathbb{B}(\mathcal{H})$ is spanned by cp maps.

Proof

Lemma 8.3 together with generalised Stinespring's representation theorem provides two cp maps

$$\phi_1(a) = V_1^*\pi(a)V_1, \quad \phi_2(a) = V_2^*\pi(a)V_2,$$

with $\|\phi_1\|_{\mathrm{cb}} = \|\phi_2\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$. Let $\phi := \frac{1}{2}(\phi_1 + \phi_2)$. By triangle inequality its cb-norm is smaller equal than ϕ, and it is clearly cp. Notice that

$$\begin{aligned} 2(\psi + \operatorname{Re}\phi) &= \phi_1 + \phi_2 + \phi + \phi^* \\ &= V_1^*\pi V_1 + V_2^*\pi V_2 + V_1^*\pi V_2 + V_2^*\pi V_1 \\ &= (V_1 + V_2)^*\pi(V_1 + V_2), \end{aligned}$$

hence it is cp. Similarly,

$$\begin{aligned} 2(\psi - \operatorname{Re}\phi) &= (V_1 - V_2)^*\pi(V_1 - V_2), \\ 2(\psi + \operatorname{Im}\phi) &= (V_1 - iV_2)^*\pi(V_1 - iV_2), \\ 2(\psi - \operatorname{Im}\phi) &= (V_1 + iV_2)^*\pi(V_1 + iV_2) \end{aligned}$$

are all cp maps. Eventually, notice that

$$\begin{aligned} \phi &= \operatorname{Re}\phi + i\operatorname{Im}\phi \\ &= \frac{1}{2}(\operatorname{Re}\phi + \psi) + \frac{1}{2}(\operatorname{Re}\phi - \psi) + \frac{i}{2}(\operatorname{Im}\phi + \psi) + \frac{i}{2}(\operatorname{Im}\phi - \psi). \end{aligned}$$

So ϕ is spanned by cp maps $A \to \mathbb{B}(\mathcal{H})$.

Remark

Wittstock's decomposition theorem can be viewed as an analogue of the decomposition theorem of elements in a C*-algebra by positive elements. But the result is only true for mapping into **injective** C*-algebras, because we are making essential use of Averson's extension theorem. This does not hold for non-injective C*-algebras. Smith [Smi83] constructed a cb map into $C([0,1])$ which is not spanned by cp maps.

2 Bimodule maps and applications to Schur multipliers

Let $A \in \mathbb{M}_n$. The Schur multiplier S_A is the linear map

$$\mathbb{M}_n \to \mathbb{M}_n, \quad (B_{ij}) \mapsto (A_{ij}B_{ij}).$$

But it is usually more interesting (and far more complicated) to study the case where A is an infinite matrix. This yields **infinite** Schur multipliers. Such maps are first studied by Schur [Sch11]. I did not study the history quite well, but it seems that Grothendieck [Gro56] was the first to characterise the boundedness conditions of infinite Schur multipliers. The surprising result is that a (possibly infinite) Schur multiplier is cb precisely when it is bounded, and its cb-norm equals its operator norm. The most elegant proof is due to Smith [Smi91]. We will study his techinique in the remaining part of this talk.

2.1 Bimodule map and matrical norm

Let $C \subseteq A$ be a C*-subalgebra. Then A can be viewed as a C-bimodule. If C is a C*-subalgebra of both A and B, then we say a linear map $\phi\colon A \to B$ is a C-bimodule map if

$$c\phi(a)c' = \phi(cac')$$

for all $c, c' \in C$ and $a \in A$.

The following definition is due to Smith [Smi91].

Definition

Let $C \subseteq A$ be a C*-subalgebra, turning A into a C-bimodule. We say A is **matrically-normed** (as a C-bimodule), if for any $(a_{ij}) \in \mathbb{M}_n(A)$, the following equation holds:

$$\|(a_{ij})\| = \sup\{\|\sum_{i,j} x_i a_{ij} y_j\| \mid x_i, y_j \in C,\ \|\sum_i x_i x_i^*\| \le 1,\ \|\sum_j y_j^* y_j\| \le 1\}.$$

The definition can be rewritten in a more condensed way. Consider A^n as the Hilbert A-module with the canonical A-value inner product. Then $a = (a_{ij}) \in \mathbb{M}_n(A) \cong \mathrm{End}_A(A^n)$ and the equation above can be written as

$$\|a\| = \sup\{\|\langle x, ay\rangle\| \mid x, y \in C^n,\ \|x\| \le 1,\ \|y\| \le 1\}.$$

Notice that $\|\langle x, ay\rangle\| \le \|x\|\|a\|\|y\| \le \|a\|$. So A is matrically normed if and only if all these equalities hold.

For bimodule maps into matrically-normed C*-algebra, we have the following nice theorem:

Theorem 8.6

If C is a C*-subalgebra of both A and B, and B is matrically-normed as a C-bimodule. Then any bounded C-bimodule map $\phi\colon A \to B$ is cb.

Proof

A straightforward computation shows:

$$\begin{aligned}\|\phi_n(a_{ij})\| &= \sup\|\sum_{i,j} x_i \phi(a_{ij}) y_j\| \\ &= \sup\|\phi(\sum_{i,j} x_i a_{ij} y_j)\| \\ &\le \|\phi\|\|a\|.\end{aligned}$$

So $\|\phi\|_{\text{cb}} \leq \|\phi\|$. This implies $\|\phi\|_{\text{cb}} = \|\phi\|$.

Another way is to identify ϕ_n with $\phi \otimes \text{id}$ acting on $A \otimes \mathbb{M}_n$, and the proof is immediate.

Example

Any linear map is a $\mathbb{C}$-bimodule map. Let X be compact Hausdorff. Then $\mathrm{C}(X)$ is a $\mathbb{C}$-bimodule. We claim that it is matrically-normed: consider the map

$$\mathbb{M}_n(\mathrm{C}(X)) \to \mathbb{R}_{\geq 0}, \quad (f_{ij}) \mapsto \|(f_{ij}(x))\|.$$

Since X is compact, there exists $x_0 \in X$ such that $\|f_{ij}(x_0)\| = \|f_{ij}\|$. So we can find $x, y \in \mathbb{C}^n$ with $\|x\| \leq 1$ and $\|y\| \leq 1$, so that (f_{ij}) reaches its supremum. This implies:

Proposition

If B is commutative. Then any bounded linear map $A \to B$ is cb.

2.2 Finite Schur multiplier

Let $A \in \mathbb{M}_n$. The finite Schur multiplier S_A is the linear map

$$\mathbb{M}_n \to \mathbb{M}_n, \quad (B_{ij}) \mapsto (A_{ij} B_{ij}).$$

We have the following theorem:

Theorem 8.7

Let $A \in \mathbb{M}_n$. The followings are equivalent:

1. $\|S_A\| \leq 1$.
2. $\|S_A\|_{\text{cb}} \leq 1$.
3. There exists a Hilbert space $\mathcal{H}$, $x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{H}$, such that

$$\|x_i\| \leq 1, \ \|y_j\| \leq 1, \text{ and } \langle x_i, y_j \rangle = A_{ij}.$$

Proof (part)

For (1) $\Rightarrow$ (2), we do not give the proof, but claim that $\mathbb{M}_n$ is a $\mathbb{D}_n$-bimodule, $S_A \colon \mathbb{M}_n \to \mathbb{M}_n$ is a $\mathbb{D}_n$-bimodule map and $\mathbb{M}_n$ is **matrically-normed** as a $\mathbb{D}_n$-bimodule. Here $\mathbb{D}_n$ denotes the C^*-subalgebra of diagonal $n \times n$-matrices. If this is done, then the previous theorem shows that $\|S_A\|_{\text{cb}} = \|S_A\|$. For the technical proof see Paulsen [Pau02].

(2) $\Rightarrow$ (3): by generalised Stinespring's representation theorem, there are a representation $\pi \colon \mathbb{M}_n \to \mathbb{B}(\mathcal{H})$ and bounded operators $V_1, V_2 \colon \mathbb{C}^n \to \mathcal{H}$ such that

$$S_A(B) = V_1^* \pi(B) V_2$$

for all $B \in \mathbb{M}_n$. Notice that for $B = E_{ij}$, the matrix with the only non-zero entry with value 1 at (i, j), then

$$A_{ij} = S_A(E_{ij}) = V_1^* \pi(E_{ij}) V_2.$$

Now we have a simple construction:

$$\begin{aligned}A_{ij} &= \langle e_i, A_{ij} e_j\rangle \\ &= \langle e_i, V_1^* \pi(E_{ij}) V_2 e_j\rangle \\ &= \langle V_1 e_i, \pi(E_{i1}E_{1j}) V_2 e_j\rangle \\ &= \langle \pi(E_{1i}) V_1 e_i, \pi(E_{1j}) V_2 e_j\rangle \\ &=: \langle x_i, y_j\rangle.\end{aligned}$$

Obviously $\|x_i\| \leq 1$ and $\|y_j\| \leq 1$.

(3) ⇒ (1): the proof is the same with that in Paulsen's book, but I write in a different form. We have

$$\begin{aligned}\|S_A(B)\| &= \sup \|\sum_{ij} \alpha_i (S_A(B))_{ij} \beta_j\| \\ &= \sup \|\sum_{i,j} \alpha_i A_{ij} B_{ij} \beta_j\| \\ &= \sup \|\sum_{i,j} \alpha_i \langle x_i, y_j\rangle B_{ij} \beta_j\| \\ &= \sup \|\sum_{i,j} (\alpha_i \otimes x_i)(B_{ij} \otimes \mathrm{id})(\beta_j \otimes y_j)\| \\ &\leq \|\alpha \otimes x\| \|B \otimes \mathrm{id}\| \|\beta \otimes y\| \\ &\leq \|\alpha\| \|B\| \|\beta\| \leq \|B\|.\end{aligned}$$

So $\|S_A\| \leq 1$.

2.3 Infinite Schur multiplier

For the infinite case, we need to restrict to **bounded** Schur multipliers $S_A : \mathbb{B}(\ell^2) \to \mathbb{B}(\ell^2)$. The boundedness condition is far more complicated. Fortunately, a similar theorem as in the finite case holds:

Theorem 8.8

Let A be an infinite matrix. Then the followings are equivalent:

1. S_A defines a bounded operator $\mathbb{B}(\ell^2) \to \mathbb{B}(\ell^2)$, and $\|S_A\| \leq 1$.
2. S_A defines a completely bounded operator $\mathbb{B}(\ell^2) \to \mathbb{B}(\ell^2)$, and $\|S_A\|_{\mathrm{cb}} \leq 1$.
3. There exists a Hilbert space $\mathcal{H}$, $\{x_i\}_{i=1}^{\infty}, \{y_j\}_{j=1}^{\infty} \subseteq \mathcal{H}$, such that

$$\|x_i\| \leq 1,\ \|y_j\| \leq 1, \text{ and } \langle x_i, y_j\rangle = A_{ij}.$$

Proof (part)

We only prove (1) ⇒ (2) because (2) ⇒ (3) and (3) ⇒ (1) are essentially the same with the finite case. Notice that for $B \in \mathbb{B}(\ell^2)$:

$$\|B\| = \sup_n \|B_n\|,$$

where $B_n = (B_{ij})_{i,j=1}^n$ is the finite-dimensional cut-off of B.

We claim that $\|S_A\| = \sup_n \|S_{A_n}\|$. Clearly $\|S_A\| \geq \sup_n \|S_{A_n}\|$ because $\|S_A\| \geq \|S_{A_n}\|$ for all n. Conversely, we have

$$
\begin{aligned}
\|S_A(B)\| &= \sup_n \|S_A(B)_n\| = \sup_n \|S_{A_n}(B)\| \\
&\le \sup_n \|S_{A_n}\| \|B\| = \sup_n \|S_{A_n}\| \cdot \|B\|.
\end{aligned}
$$

Therefore $\|S_A\| = \sup_n \|S_{A_n}\|$. A similar result holds for cb-norms as well. Now use the fact that finite Schur multipliers have the same norm and cb-norm, we obtain

$$
\|S_A\| = \sup_n \|S_{A_n}\| = \sup_n \|S_{A_n}\|_{\text{cb}} = \|S_A\|_{\text{cb}},
$$

and the proof (1) $\Rightarrow$ (2) is done.

In the end, we provide a criterion for boundedness of infinite Schur multipliers, based on the previous theorem.

Theorem 8.9

Let $A = (A_{ij})$ be an infinite matrix. Let $B = (B_{ij})$ where

$$
B_{ij} := A_{i,j} + A_{i+1,j+1} - A_{i,j+1} - A_{i+1,j}.
$$

If the followings hold:

- $A_{\infty,j} := \lim_i A_{i,j}$ exists for all j.
  $A_{i,\infty} := \lim_j A_{i,j}$ exists for all i.
- $\lim_i \lim_j A_{i,j}$ and $\lim_j \lim_i A_{i,j}$ both exist.
- $\{B_{ij}\}_{i,j}$ is absolutely summable.

Then S_A is bounded, hence cb.

Proof

Since $\{B_{ij}\}$ is absolutely summable. We have

$$
\sum_{j=\beta}^{\infty} \sum_{i=\alpha}^{\infty} B_{ij} = A_{\alpha,\beta} - A_{\alpha,\infty} - A_{\infty,\beta} + \lim_j \lim_i A_{i,j},
$$

and

$$
\sum_{i=\alpha}^{\infty} \sum_{j=\beta}^{\infty} B_{ij} = A_{\alpha,\beta} - A_{\alpha,\infty} - A_{\infty,\beta} + \lim_i \lim_j A_{i,j}.
$$

So $\lim_j \lim_i A_{i,j} = \lim_i \lim_j A_{i,j}$. Define it to be $A_{\infty,\infty}$.

Now let $\mathcal{H}$ be the Hilbert space spanned by orthonormal basis $\{e, f, g\} \cup \{e_{ij}\}_{i,j=1}^{\infty}$. Define

$$
\begin{aligned}
x_i &:= e + f + A_{i,\infty} g + \sum_{\alpha=1}^{\infty} \sum_{\beta=i}^{\infty} d_{\alpha,\beta} e_{\alpha,\beta}, \\
y_j &:= -A_{\infty,\infty} e + A_{\infty,j} f + g + \sum_{\alpha=j}^{\infty} \sum_{\beta=1}^{\infty} c_{\alpha,\beta} e_{\alpha,\beta}.
\end{aligned}
$$

where $c_{\alpha\beta}$ and $d_{\alpha\beta}$ are defined such that $|c_{\alpha\beta}| = |d_{\alpha\beta}|$ and $c_{\alpha\beta} \bar{d}_{\alpha\beta} = b_{\alpha\beta}$. One can check that $\{x_i\}$

and $\{y_j\}$ are uniformly bounded, and

$$\langle x_i, y_j \rangle = -A_{\infty,\infty} + A_{\infty,j} + A_{i,\infty} + \sum_{\alpha=i}^{\infty} \sum_{\beta=j}^{\infty} B_{\alpha,\beta} = A_{i,j}.$$

By Theorem 8.8, S_A is bounded.

References

[Gro56] Alexandre Grothendieck. *Résumé de la théorie métrique des produits tensoriels topologiques.* Soc. de Matemática de São Paulo, 1956.

[Pau84] Vern Paulsen. Every completely polynomially bounded operator is similar to a contraction. *Journal of functional analysis*, 55(1):1–17, 1984.

[Pau02] Vern Paulsen. *Completely bounded maps and operator algebras.* Number 78. Cambridge University Press, 2002.

[Sch11] Jssai Schur. Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. 1911.

[Smi83] Roger R Smith. Completely bounded maps between C*-algebras. *Journal of the London Mathematical Society*, 2(1):157–166, 1983.

[Smi91] Roger R Smith. Completely bounded module maps and the Haagerup tensor product. *Journal of Functional Analysis*, 102(1):156–175, 1991.